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# Spring 2013 Math 152

**Overview of Material for Test III**

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## Integral Test, Comparison Tests

1. **Integral Test:** If $f(x)$ is a positive, continuous, decreasing function on $[k,\infty]$, where $k$ is a non-negative integer, and $a_n = f(n)$. Then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)\,dx$ either both converge or both diverge.

   "If the improper integral converges, so does the series. If the improper integral diverges, so does the series".

   TIP: Use the integral test if the terms of the series are positive, decreasing and integrable.

2. **Comparison Test:** Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ are series of **positive terms**.

   - If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ is also convergent. "If the larger series converges, so does the smaller series".
   - If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ is also divergent. "If the smaller series diverges, so does the larger series".

   TIP: Use the comparison test if the terms of the series are positive and are comparable to a p-series or geometric series.

3. **Limit Comparison Test:** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

   TIP: Use the limit comparison test if the terms of the series are positive and are comparable to a p-series or geometric series, however the inequality may not point in the right direction for the comparison test to be conclusive.

4. **Remainder Estimate**: Suppose $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$ is the $n^{th}$ partial sum of the convergent series $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$. Then the **remainder** in using $s_n$ to approximate the sum $S$ is defined to be

   $R_n = S - s_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + ....$

   Moreover, if $\sum_{n=1}^{\infty} a_n$ was shown to be convergent by the integral test or a comparison test (where $a_n = f(n)$), then

   $R_n = \sum_{i=n+1}^{\infty} a_i \le \int_n^{\infty} f(x)\,dx.$

## Alternating series, Ratio Test and Remainders

5. **The Alternating Series Test**: The alternating series $\sum_{n=k}^{\infty} (-1)^n a_n$, where $a_n > 0$, converges if it satisfies both conditions given below:

   - $a_{n+1} \le a_n$ (ie the sequence $\{a_n\}$ is decreasing).
   - $\lim_{n\to\infty} a_n = 0$

   TIP: Use the alternating series test if the series is an alternating series which fails the test for divergence.

6. **Def:** A series is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series is **conditionally convergent**. To test for absolute convergence, you either use the Ratio Test (see conditions below) or test $\sum_{n=1}^{\infty} |a_n|$ for convergence, which usually involves the p - series test, integral test or comparison test. Note: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

7. **The Ratio Test**:

   - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
   - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
   - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test fails (ie is inconclusive).

   Tip: Use the ratio test if the series contains either or a factorial or an exponential (or both).

8. **Remainder Estimate and The Alternating Series Theorem**

   If $\sum_{n=1}^{\infty} (-1)^n a_n$, $a_n > 0$, is a convergent alternating series, and a partial sum

   $s_n = \sum_{i=1}^{n} (-1)^i a_i$ is used to approximate the sum of the series with remainder $R_n$, then $|R_n| \le a_{n+1}$.

## Power Series

9. **Def:** A **Power Series** is a series of the form $\sum_{n=1}^{\infty} c_n(x-a)^n$, where $x$ is the variable and the $c_n$'s are called the coefficients of the series. More generally, $\sum_{n=1}^{\infty} c_n(x-a)^n$ is called a power series *centered* at $x = a$, or a power series *about* $a$. Specifically, $\sum_{n=1}^{\infty} c_n x^n$ is a power series centered at zero.

   Moreover, the set of all values of $x$ for which the series converges is called **the interval of convergence**, denoted by $I$. The **radius of convergence** is $R = \frac{1}{2}$ of the length of $I$. "The radius of convergence is the maximum distance you deviate from the center to have convergence".

10. **Theorem:** For a given power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ there are only three possibilities:

   (i) The series converges only for $x = a$, in which case $I = \{a\}$ and $R = 0$.

   (ii) The series converges for all $x$, in which case $I = (-\infty, \infty)$ and $R = \infty$.

   (iii) There is a positive number $R$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$, in which case $I = (a-R, a+R)$ (test the end points for convergence) and the radius of convergence is $R$. In order to find the radius of convergence, apply the ratio test.

## Representing Functions as Power Series

11. If $|x| < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

12. **Theorem:** If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence $R$, then

   a.) $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$ and has a radius of convergence $R$.

   b.) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$ and has a radius of convergence $R$.

13. To find a power series representation for a function of the form $\ln(ax+b)$, first take the derivative of the function, then integrate its power series representation.

14. To find a power series representation for a function of the form $\frac{1}{(1+x)^2}$, first integrate the function, then take the derivative of its power series representation.

## Taylor and Maclaurin Series

15. The **Taylor Series** of $f(x)$ at $x = a$ is

   $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
   $$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

16. The **Maclaurin series** is the Taylor series about 0, that is

   $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$
   $$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

17. Known Maclaurin series. Have these Maclaurin series memorized and know when to use them.

   (a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \ldots$, provided $-1 < x < 1$.

   (b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$, provided $-\infty < x < \infty$.

   (c) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$, provided $-\infty < x < \infty$.

   (d) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$, provided $-\infty < x < \infty$.

   (e) $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$, provided $-1 < x < 1$.

## Taylor Polynomials

18. Let $f(x)$ be a function. The $n^{th}$ degree **Taylor Polynomial** for $f(x)$ at $x = a$ is

   $$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

   where $f^{(i)}(a)$ is the $i^{th}$ derivative of $f(x)$ at $x = a$.

19. **Taylor's Inequality:** An upper bound on the absolute value of the remainder is

   $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

   where $|f^{(n+1)}(x)| \leq M$ for $x$ in an interval containing $a$. Note: This formula will be provided on the exam.

## Three dimensional Coordinate System

20. The distance from the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is
$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

21. The equation of the sphere with center $(h, k, l)$ and radius $r$ is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

## Vectors and the Dot Product

22. The Algebra of Vectors: Suppose $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$ are vectors and $c$ is a scalar.

a.) Scalar Multiplication: $c\overrightarrow{a} = \langle ca_1, ca_2, ca_3 \rangle$.

b.) Vector Sum: $\overrightarrow{a + b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$ .

c.) Vector Difference: $\overrightarrow{a - b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$ .

d.) Vector Length: $|\overrightarrow{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$.

e.) Unit Vector: A unit vector in the direction of $\overrightarrow{a}$ is $\overrightarrow{u} = \dfrac{\overrightarrow{a}}{|\overrightarrow{a}|}$

f.) Dot Product: $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}| \cos\theta$, where $\theta$ is the angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

23. Alternative Dot Product: If you know the components of $\overrightarrow{a}$ and $\overrightarrow{b}$, then $\overrightarrow{a} \cdot \overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3$.

24. The angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is
$\theta = \arccos\left(\dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}\right)$

25. Vector and Scalar Projections
• The **Vector Projection** of $\overrightarrow{b}$ onto $\overrightarrow{a}$, also called the projection of $\overrightarrow{b}$ in the direction of $\overrightarrow{a}$, is:

$\overrightarrow{proj_a b} = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|^2}\overrightarrow{a}$

• The **Scalar Projection** of $\overrightarrow{b}$ onto $\overrightarrow{a}$ (also called the scalar component of $\overrightarrow{b}$ onto $\overrightarrow{a}$) is:

$comp_a b = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|}$