Section 3.9

Derivatives of Parametric Curves: If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. *

This gives us a way to find the slope of the tangent line to the parametric curve at $t = t_0$: $m = \frac{dy}{dx}\Big|_{t=t_0}$.

1. Given $x = \cos t$ and $y = t^2$, find $\frac{dy}{dx}$. Next, find the equation of the tangent line at $t = \frac{\pi}{4}$.

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-\sin t}}$$

① point: $t = \frac{\pi}{4}$ — $x = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $y = \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16}$

$\boxed{\text{point: } \left(\frac{\sqrt{2}}{2}, \frac{\pi^2}{16}\right)}$

② $m = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}}$

$m = \frac{2t}{-\sin t}\Big|_{t=\frac{\pi}{4}}$ → $m = \frac{\frac{\pi}{2}}{-\frac{\sqrt{2}}{2}}$

$m = \frac{\pi}{2}\left(-\frac{2}{\sqrt{2}}\right)$

$\boxed{m = \frac{-\pi}{\sqrt{2}}}$

$\boxed{\text{tangent line: } y - \frac{\pi^2}{16} = \frac{-\pi}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right)}$

2. Let $x = t^4 - 4t^3$ and $y = 3t^2 - 6t$.

a.) Find the equation of the tangent line at the point $(5, 9)$.

① Find t so that $x = 5$, $y = 9$

$t^4 - 4t^3 = 5 \rightarrow t = -1$

$3t^2 - 6t = 9 \rightarrow t = -1$

$t = -1$

$m = \frac{dy/dt}{dx/dt}\Big|_{t=-1} = \frac{6t-6}{4t^3 - 12t^2}\Big|_{t=-1}$

$m = \frac{-12}{-16} = \frac{3}{4}$

point $= (5, 9)$

$m = \frac{3}{4}$

equation $y - 9 = \frac{3}{4}(x - 5)$

b.) Find all point(s) on the curve where the tangent line is vertical or horizontal.

vertical → undefined slope

horizontal → zero slope

$\begin{cases} x = t^4 - 4t^3 \\ y = 3t^2 - 6t \end{cases}$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

① horizontal tangents

$m = 0 \rightarrow \frac{dy}{dt} = 0$

$6t - 6 = 0$

$t = 1$ — $x = -3$, $y = -3$

horizontal tangent occurs at $(-3, -3)$

② vertical tangents $m =$ undefined

$\frac{dx}{dt} = 0$

$4t^3 - 12t^2 = 0$

$4t^2(t - 3) = 0$

$t = 0, \ t = 3$

$t = 0$ — $x = 0$, $y = 0$

$t = 3$ — $x = -27$, $y = 9$

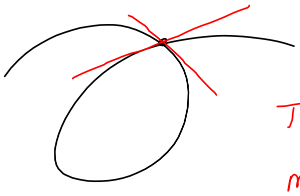
vertical tangents at $(0, 0)$ & $(-27, 9)$

3. Show the curve $x = \cos t$ and $y = \sin t \cos t$ has two tangents at $(0, 0)$. Find the equations of these tangent lines.

$x = \cos t$
$y = \sin t \cos t$

$t = \frac{\pi}{2}$ gives $x = 0$ & $y = 0$ since $\cos\frac{\pi}{2} = 0$

$t = \frac{3\pi}{2}$ gives $x = 0$ & $y = 0$



Tangent line one:

$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{2}} = \left.\frac{-\sin^2 t + \cos^2 t}{-\sin t}\right|_{t=\frac{\pi}{2}} = \frac{-1}{-1} = 1$$

$y = \sin t \cos t$

$$\frac{dy}{dt} = \sin t(-\sin t) + \cos t \cos t = -\sin^2 t + \cos^2 t$$

$\boxed{m = 1 \quad \text{point: } (0,0) \quad \text{equation: } y = x}$

Tangent line two:

$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{3\pi}{2}} = \left.\frac{-\sin^2 t + \cos^2 t}{-\sin t}\right|_{t=\frac{3\pi}{2}} = \frac{-1}{1} = -1$$

$m = -1$
point: $(0\ 0)$
equation: $y = -x$

4. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent line parallel to the line with equations $x = -7t$, $y = 12t - 5$?

solve: $\frac{1}{12} \cdot \frac{12t}{3t^2+4} = \frac{12}{-7} \cdot \frac{1}{12}$

$$\frac{t}{3t^2+4} = -\frac{1}{7}$$

$-7t = 3t^2 + 4$

$0 = 3t^2 + 7t + 4$

$0 = (3t + 4)(t + 1)$

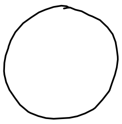
$t = \frac{-4}{3}$: $x = \frac{-208}{27}$, $y = \frac{32}{3}$

$t = -1$: $x = -5$, $y = 6$

$\boxed{\text{Answer} = \left(\frac{-208}{27}, \frac{32}{3}\right) \ \& \ (-5, 6)}$

Section 3.10

5. Water leaking onto a floor creates a circular pool with an area that increases at a rate of 3 square inches per minute. How fast is the radius of the pool increasing when the radius is 10 inches?



r = radius

A = area

Given: $\frac{dA}{dt} = 3$

Find: $\left.\frac{dr}{dt}\right|_{r=10}$

$A = \pi r^2$

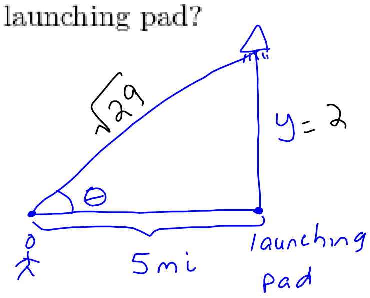
$\cancel{\frac{dA}{dt}}^{\,3} = 2\pi \cancel{r}^{\,10} \frac{dr}{dt}$

$3 = 2\pi(10)\frac{dr}{dt}$

$$\boxed{\frac{dr}{dt} = \frac{3}{20\pi} \ \frac{\text{in}}{\text{min}}}$$

6. When a rocket is 2 miles high, it is moving vertically upward at a speed of 300 mph. At that instant, how fast is the angle of elevation of the rocket increasing, as seen by an observer on the ground 5 miles from the launching pad?

when $y = 2$



given: $\left.\frac{dy}{dt}\right|_{y=2} = 300$ mph

Find $\frac{d\theta}{dt}$ at this instant

$\tan\theta = \frac{y}{5}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{\cancel{dy/dt}^{\,300}}{5}$

$\frac{dy}{dt} = 300$

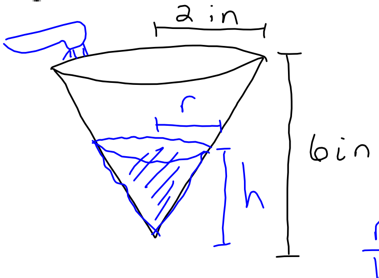
$\cos\theta = \frac{5}{\sqrt{29}}$

$\cos^2\theta = \frac{25}{29}$

$\frac{d\theta}{dt} = \frac{300}{5}\cos^2\theta$

$$\boxed{\frac{d\theta}{dt} = \frac{300}{5}\cdot\frac{25}{29} \ \frac{\text{rad}}{\text{hour}}}$$

7. A filter in the shape of a cone is 6 inches high and has a radius of 2 inches at the top. A solution is poured in the cone so that the water level is rising at a rate of $\frac{3}{2}$ inches per second. How fast is the water being poured in when the water level has a depth of 2 inches?.



given: $\frac{dh}{dt} = \frac{3}{2}$

Find: $\left.\frac{dV}{dt}\right|_{h=2}$

$\frac{r}{h} = \frac{2}{6}$

$r = \frac{1}{3}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} 3 h^2 \frac{dh}{dt}$ ($h = 2$, $\frac{dh}{dt} = \frac{3}{2}$)

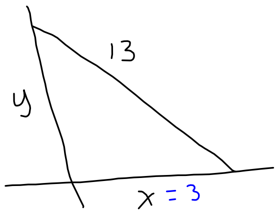
$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3 \cdot 4 \cdot \frac{3}{2}$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 9 \cdot 2$

$= \frac{\pi}{3} \cdot 2$

$\boxed{\frac{dV}{dt} = \frac{2\pi}{3} \frac{in^3}{sec}}$

8. One end of a 13 foot ladder is on the ground, and the other end rests on a vertical wall. If the top of the ladder is being pushed up the wall at a rate of 1 foot per second, how fast is the base of the ladder approaching the wall when it is 3 feet from the wall?



$\frac{dy}{dt} = 1$

Find $\left.\frac{dx}{dt}\right|_{x=3}$

$y = \sqrt{169-9} = \sqrt{160}$

$x^2 + y^2 = 169$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(3)\frac{dx}{dt} + 2\sqrt{160}\,(1) = 0$

$6\frac{dx}{dt} = -2\sqrt{160}$

$\boxed{\frac{dx}{dt} = \frac{-2\sqrt{160}}{6} \frac{f}{s}}$

9. A point moves around the circle $x^2 + y^2 = 9$. When the point is at $(-\sqrt{3}, \sqrt{6})$, its x coordinate is increasing at a rate of 20 units per second. How fast is its y coordinate changing at that instant?

$$\left.\frac{dx}{dt}\right|_{\substack{x=-\sqrt{3}\\ y=\sqrt{6}}} = 20$$

Find $\frac{dy}{dt}$

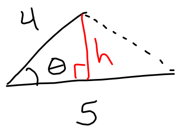
$$x^2 + y^2 = 9$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(-\sqrt{3})(20) + 2\sqrt{6}\frac{dy}{dt} = 0$$

$$2\sqrt{6}\frac{dy}{dt} = 40\sqrt{3} \rightarrow \frac{dy}{dt} = \frac{40\sqrt{3}}{2\sqrt{6}}\ \frac{\text{units}}{\text{sec}}$$

10. Two sides of a triangle have lengths 5 m and 4 m. The angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is 60°.



given: $\frac{d\theta}{dt} = 0.06$

Find $\left.\frac{dA}{dt}\right|_{\theta=60^\circ}$

h= height

$\boxed{b=5}$

$\sin\theta = \frac{h}{4}$

$\boxed{h = 4\sin\theta}$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(5)(4)\sin\theta$$

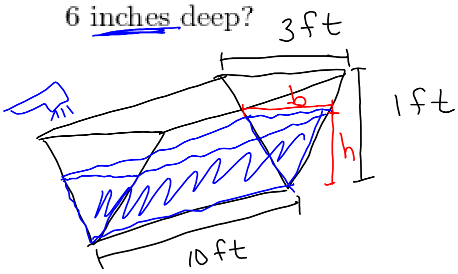
$$A = 10\sin\theta$$

$$\frac{dA}{dt} = 10\cos\theta\frac{d\theta}{dt}$$

$$\frac{dA}{dt} = 10\cdot\cos 60^\circ(0.06)$$

$$= \boxed{(10)\left(\frac{1}{2}\right)(0.06)\ \frac{m^2}{sec}}$$

11. A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across the top and have a height of 1 foot. If the trough is filled with water at a rate of 12 cubic feet per minute, how fast is the water level rising when the water is 6 inches deep?



given: $\frac{dV}{dt} = 12 \frac{ft^3}{min}$

Find $\left.\frac{dh}{dt}\right|_{h = 6\text{ in}}$

$h = \frac{1}{2}$ ft

$\frac{b}{h} = \frac{3}{1}$

$b = 3h$

$V = \left(\frac{1}{2} bh\right)(10)$

$V = \frac{1}{2}(3h)h(10)$

$V = 15h^2$

$\cancel{\frac{dV}{dt}}^{\,12} = 30\cancel{h}^{\,\frac{1}{2}} \frac{dh}{dt}$

$12 = 15 \frac{dh}{dt}$

$\frac{12}{15} = \frac{dh}{dt}$

$\boxed{\frac{4}{5} \frac{ft}{min} = \frac{dh}{dt}}$

Section 3.11

12. Let $y = 4 - x^2$. Find Δy if x changes from $x = 1$ to $x = 1.5$.

$f(x) = y = 4 - x^2$ x changes from $x=1$ to $x = 1.5 = \frac{3}{2}$

$\Delta x = \frac{1}{2}$

$\Delta y = f\left(\frac{3}{2}\right) - f(1)$

$\Delta y = 4 - \frac{9}{4} - (3)$

$\Delta y = 1 - \frac{9}{4} = \frac{-5}{4}$

13. If $f(x) = 4 - x^2$, find dy if $x = 1$ and $dx = \frac{1}{2}$.

dy = differential

if $y = f(x)$, then $dy = f'(x)\,dx$

$dy = f'(1)\left(\frac{1}{2}\right)$

$= -2\left(\frac{1}{2}\right) = -1$

$f(x) = 4 - x^2$

$f'(x) = -2x$

if $x = 1$, $f'(1) = -2$

$dx = \frac{1}{2}$

14. Find the differential dy if $y = \dfrac{r}{r+1}$ and $dr = 0.5$.

$f(r) = y = \frac{r}{r+1}$, $dy = f'(r)\,dr$

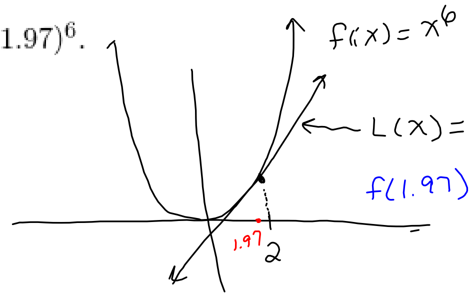
$f'(r) = \dfrac{(1)(r+1) - r(1)}{(r+1)^2}$

$f'(r) = \dfrac{1}{(r+1)^2}$

$dy = \dfrac{1}{(r+1)^2}\,dr$

$$\boxed{dy = \frac{1}{(r+1)^2}(.5)}$$

15. Use differentials to approximate $(1.97)^6$.



$L(x) = 64 + 192(x-2)$

$f(1.97) \approx L(1.97)$

$\approx 64 + 192(1.97-2)$

$\approx 64 + 192(-0.03)$

≈ 58.24

$f(x) = x^6$

approximate $f(1.97)$

linear approximation at $x = 2$ is $L(x) = f(2) + f'(2)(x-2)$

$f(x) = x^6$ $\qquad f'(x) = 6x^5$

$f(2) = 64$ $\qquad f'(2) = 6(32)$

$f'(2) = 192$

$L(x) = 64 + 192(x-2)$

second method: use differentials

$f(a+dx) \approx f(a) + f'(a)\,dx$

$f(1.97) \approx \cancel{f(2)}^{64} + \cancel{f'(2)}^{192}(-0.03)$

$f(x) = x^6$

$a = 2$

$dx = -0.03$

16. Use differentials to approximate $\cos(31.5°)$

$f(x) = \cos x$

$a = 30°$

$dx = 1.5°$

$f(31.5°) \approx \cancel{f(30°)}^{\frac{\sqrt{3}}{2}} + \cancel{f'(30°)}^{-\frac{1}{2}}(1.5°)$

$f(x) = \cos x$ $\qquad f'(x) = -\sin x$

$f(30°) = \frac{\sqrt{3}}{2}$ $\qquad f'(30°) = -\frac{1}{2}$

convert $1.5°$ to radians

$\cos(31.5°) \approx \frac{\sqrt{3}}{2} - \frac{1}{2}\left(1.5 \frac{\pi}{180}\right)$

17. Find the linear approximation for $f(x) = \frac{1}{x}$ at $x =$ 4. $a=4$

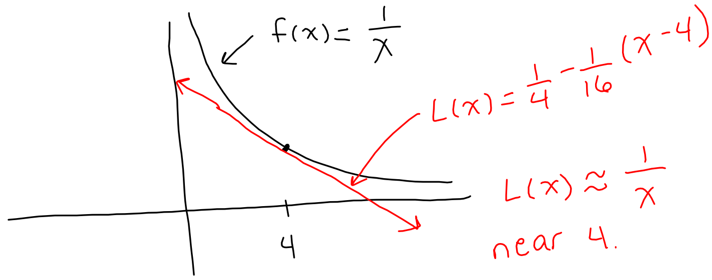
$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(4) + f'(4)(x-4)$

$$\boxed{L(x) = \frac{1}{4} - \frac{1}{16}(x-4)}$$

$f(x) = \frac{1}{x}$ $\quad f'(x) = \frac{-1}{x^2}$

$f(4) = \frac{1}{4}$ $\quad f'(4) = \frac{-1}{16}$



$L(x) \approx \frac{1}{x}$ near 4.

18. Find the linear approximation for $f(x) = \sqrt[3]{x+1}$ at $x = 0$ and use it to approximate $\sqrt[3]{0.95}$

$a=0$

① $L(x)$ at $x=0$ is

$L(x) = f(0) + f'(0)\,x$

$$\boxed{L(x) = 1 + \frac{1}{3}x}$$

$f(x) = \sqrt[3]{x+1} = (x+1)^{\frac{1}{3}}$

$f(0) = 1$

$f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$

$f'(0) = \frac{1}{3}(1)^{-\frac{2}{3}} = \frac{1}{3}$

② $1 + \frac{1}{3}x \approx \sqrt[3]{x+1}$ $\quad$ let $x = -0.05$

$$\boxed{1 + \frac{1}{3}(-0.05) \approx \sqrt[3]{.95}}$$

19. The radius of a circular disk is given to be 24 cm with a maximum error in measurement of 0.2 cm.

given: $r = 24$

$\Delta r = .2$

(a) What is the maximum error in the calculated area of the disk?

$A = \pi r^2$

$\Delta A = A(24.2) - A(24)$

$\Delta A = \pi(24.2)^2 - \pi(24)^2$

$\boxed{\Delta A = 9.64\pi \text{ cm}^2}$

(b) Use differentials to approximate the maximum error in the area of the disk.

recall: $y = f(x)$

differential $dy = f'(x)\,dx$

$A = \pi r^2$

$dA = 2\pi r\,dr$

$r = 24$

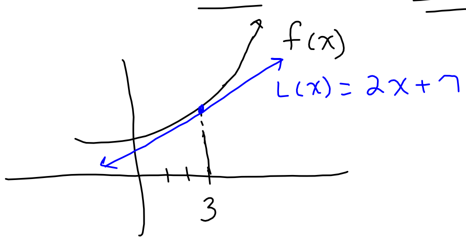
$dr = .2$

$dA = 2\pi(24)(.2)$

$dA = 9.6\pi \text{ cm}^2$

(c) What is the relative error? $= \dfrac{dA}{A} = \dfrac{9.6\pi}{\pi(24)^2}$

20. Suppose for a function f, the linear approximation for $f(x)$ at $\underline{\underline{a = 3}}$ is given by $\underline{\underline{y = 2x + 7}}$.



$f(3) = L(3)$

$f(3) = 2(3) + 7$

$f(3) = 13$

$f'(3) = L'(3)$

$f'(3) = 2$

a.) Find the value of $f'(3)$ and $f(3)$.

$\boxed{f(3) = 13}$ $\boxed{f'(3) = 2}$

b.) If $g(x) = \sqrt{f(x)}$, find the linear approximation for $g(x)$ at $a = 3$.

$m(x) = \overset{\sqrt{13}}{\cancel{g(3)}} + \overset{\frac{1}{\sqrt{13}}}{\cancel{g'(3)}}(x-3)$

$$\boxed{m(x) = \sqrt{13} + \frac{1}{\sqrt{13}}(x-3)}$$

$g(3) = \sqrt{f(3)}$

$g(3) = \sqrt{13}$

$g(x) = \sqrt{f(x)} = \left(f(x)\right)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}\left(f(x)\right)^{-\frac{1}{2}} f'(x)$

$g'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$

$g'(3) = \dfrac{f'(3)}{2\sqrt{f(3)}} = \dfrac{2}{2\sqrt{13}}$

$= \dfrac{1}{\sqrt{13}}$

21. Find the quadratic approximation for $f(x) = \cos x$ at $x = \frac{\pi}{3}$.

The quadratic approximation for $f(x)$ at $x=a$ is

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$Q(x) = \cancel{f}\left(\frac{\pi}{3}\right) + \cancel{f'}\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{\cancel{f''}\left(\frac{\pi}{3}\right)}{2}\left(x-\frac{\pi}{3}\right)^2$$

$\frac{1}{2}$ $\quad -\frac{\sqrt{3}}{2}$ $\quad \frac{1}{2}$

$f = \cos x \qquad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f' = -\sin x \qquad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f'' = -\cos x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$$Q(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\left(x-\frac{\pi}{3}\right)^2$$