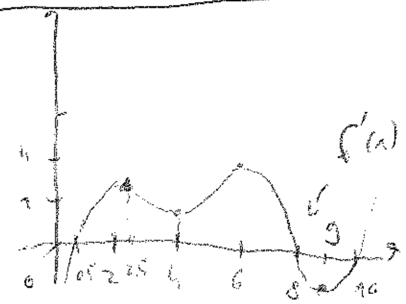
Week in Review, April 15th 2015

(1)



⇒ • f increasing for $0.5 < x < 8$ and $x > 10$
• f decreasing for $x < 0.5$ and $8 < x < 10$
• f has local maximum at 8
• f has local minima at 0.5, 10

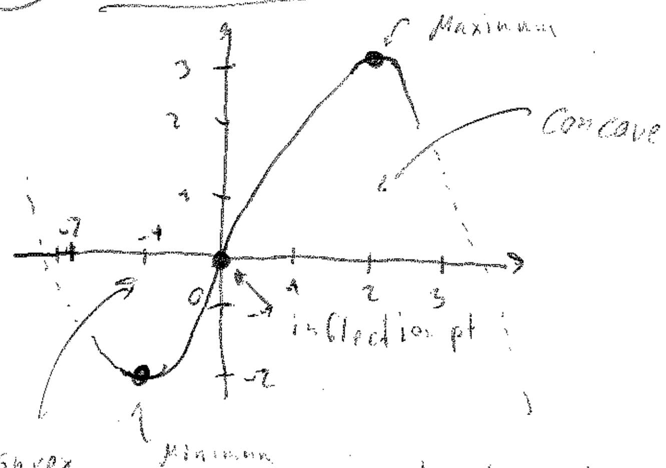
} FIRST DERIVATIVE TEST USED

• f has inflection pts at 2.5, 4, 6, 9 (extrema of $f'(x)$)

• f is convex / concave upwards for $x < 2.5$; $4 < x < 6$, $x > 9$
• f is concave / concave downwards for $2.5 < x < 4$, $6 < x < 9$.

(2) Sketch a graph!

(i) $f: \mathbb{R} \to \mathbb{R}$
(ii) $f(-1) = -2$, $f(0) = 0$, $f(2) = 3$
(iii) $f'(x) < 0$ for $x < -1$, $x > 2$
(iv) $f'(x) > 0$ for $-1 < x < 2$
(v) $f''(x) < 0$ for $x > 0$; $f''(x) > 0$ for $x < 0$



(3) Identify all critical values

Need to find:
• Roots of $f'(x)$
• Pts. where $f'(x)$ is undefined

(a) $f(x) = 4x^3 - 9x^2 - 12x + 3$

$\rightsquigarrow f'(x) = 12x^2 - 18x - 12$

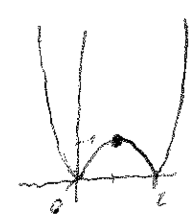
$f'(x) = 0 \iff x^2 - \frac{3}{2}x - 1 = 0 \iff x_{1,2} = \frac{3}{4} \pm \frac{1}{2}\sqrt{\frac{9}{4} + 4} = \frac{3}{4} \pm \frac{1}{2} \cdot \frac{5}{2}$

$\iff x \in \{2, -\frac{1}{2}\}$ ← critical numbers

(b) $f(x) = x^2 \cdot e^{2x} \rightsquigarrow f'(x) = 2x \cdot e^{2x} + x^2 \cdot e^{2x} \cdot 2$

$f'(x) = 0 \iff \underbrace{2x \cdot e^{2x} + x^2 \cdot e^{2x} \cdot 2}_{= 2x e^{2x} (1 + x)} = 0 \iff x = 0$ or $1 + x = 0$

⇒ crit. numb.: 0, −1



(c) $f(x) = |x^2 - 2x| = |x(x-2)|$ ~> $f'(x)$ is not def there ~> crit. numbers

~> has an apex at 0, 2

$f'(x) = \begin{cases} 2x - 2 & \text{for } x \le 0 \text{ and } x \ge 2 \\ -2x + 2 & \text{for } 0 < x < 2 \end{cases}$

⇒ $f'(x) = 0 \iff x = 1$

⇒ crit. numb. are 0, 1, 2.

d) $f(x) = (x^2 - x)^{1/3}$

$f'(x) = \frac{1}{3} \cdot (x^2 - x)^{-2/3} \cdot (2x-1) = \frac{2x-1}{3\sqrt[3]{(x^2-x)^2}}$

$\leadsto f'(x) = 0 \Leftrightarrow 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$

$\leadsto f'(x)$ is not defined for $\sqrt[3]{(x^2-x)^2} = 0 \Leftrightarrow x^2 - x = 0 \Leftrightarrow$

$x(x-1) = 0 \Leftrightarrow x = 0$ or $x = 1$

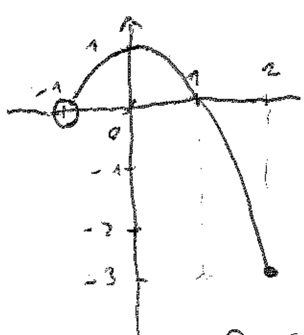
$\Rightarrow$ Critical numbers are $0, \frac{1}{2}, 1$

e) $f(x) = \frac{x+1}{x-2} \leadsto f'(x) = \frac{1 \cdot (x-2) - 1 \cdot (x+1)}{(x-2)^2} = \frac{-3}{(x-2)^2}$

$\leadsto f'(x)$ has no zeros but is undefined for $x = 2$, hence this is the only critical number.

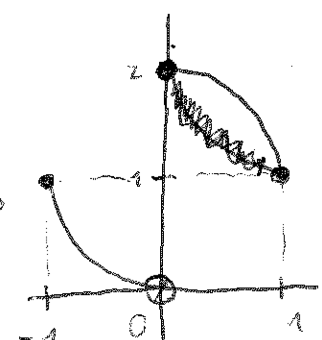
(4) Find the absolute and local extrema of the following fcts. by graphing

a) $f(x) = 1 - x^2$ for $-1 < x \leq 2$



$\leadsto$ • f has a local and global maximum at 0

• f has global minimum at 2, which is not a local minimum.



b) $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ 2 - x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$

• f has a global maximum at 0, which is no local maximum.

• f has neither global nor local minima.

(5) Find the absolute extrema.

USE: Closed Interval Method

a) $f(x) = x^3 - 12x + 1$ over $[-1, 5]$

(i) $f'(x) = 3x^2 - 12 \leadsto f'(x) = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm 2$

$f''(x) = 6x \Rightarrow f''(-2) < 0,\ f''(2) > 0$

$\Rightarrow$ f has local minimum at 2 and local maximum at -2.

(ii) $f(-1) = -1 + 12 + 1 = \underline{\underline{12}}$

$f(5) = 5^3 - 5 \cdot 12 + 1 = 125 - 60 + 1 = \underline{\underline{66}}$

(iii) f has its absolute maximum at 5 and its absolute minimum at 2.

b) $f(x) = \ln(x) \cdot x$ over $[1,3]$

(i) $f'(x) = 1 + \ln(x) \Rightarrow f'(x) = 0 \Leftrightarrow \ln(x) = -1 \Leftrightarrow x = \frac{1}{e}$

$f''(x) = \frac{1}{x} \Rightarrow f''(\frac{1}{e}) = e > 0 \Rightarrow f$ has a local minimum at $\frac{1}{e}$

BUT: $\underline{\underline{\frac{1}{e} < 1}}$.

We observe: $f'(x)$ is > 0 for all $x \in [1,3]$

$\Rightarrow f$ is strictly increasing on $[1,3]$

$\Rightarrow f$ has its global minimum at 1 and its global maximum at 3.

(6) Sketch a graph of a function satisfying the following conditions:

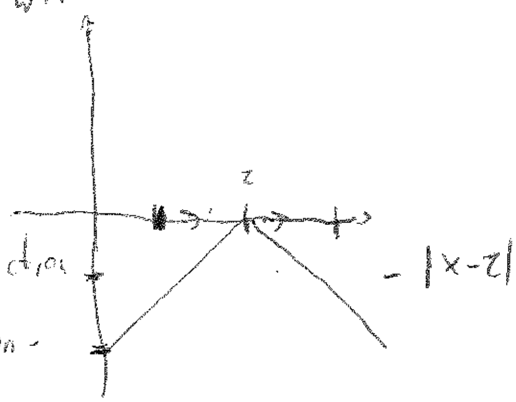
a) $x = 2$ is a critical number, but $f(x)$ has no local extrema

$\rightsquigarrow f(x) = \sqrt{x-2}$. Namely: $f'(x) = \frac{+1}{2\sqrt{x-2}}$, i.e. $x = 2$ is not defined and hence a critical number. But since $f(x)$ is only defined for $x \geq 2$, $x = 2$ cannot be local extremum. Moreover, $f'(x)$ is nowhere zero.

Hence f has no local extrema.

b) $f(x)$ is continuous function with local maximum at $x = 2$ but $f(x)$ is not differentiable at $x = 2$

$\rightsquigarrow f(x) = -|x-2|$

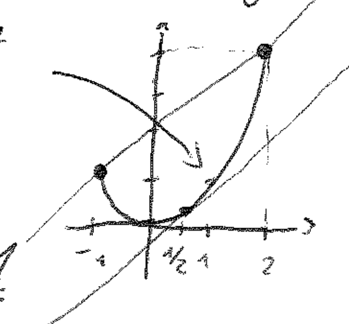


- f is continuous as absolute value function
- f has apex $x = 2$, i.e. it is not differentiable at $x = 2$
- $f(x) < 0$ for all $x \neq 2$ but $f(2) = 0$

$\Rightarrow f$ has global and local maximum at $x = 2$

(7) Mean Value Theorem: Let f be differentiable function on $[a,b]$. Then there exists a $c \in [a,b]$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $f(x) = x^2$. Verify that the MVT is satisfied for f on $[-1,2]$



$\rightsquigarrow$ We have $\frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 1}{3} = \underline{\underline{1}}$

$f'(x) = 2x$. We have for $\frac{1}{2} \in [-1,2]$: $f'(\frac{1}{2}) = \underline{\underline{1}}$

(8) Find the intervals where the fct. is in-/decreasing and all local extrema.

a) $f(x) = 3x^4 + 4x^3 - 12x^2 + 8$

$\rightsquigarrow f'(x) = 12x^3 + 12x^2 - 24x$. Hence, $f'(x) = 0 \iff$

$x = 0$ or $x^2 + x - 2 = 0 \iff x_{1/2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+8} = -\frac{1}{2} \pm \frac{3}{2}$

$\Rightarrow x \in \{-2, 1\}$

$\boxed{\text{USE: First Derivative Test}}$

- $f'(x) < 0$ for $x < -2$ and $0 < x < 1$
- $f'(x) > 0$ for $-2 < x < 0$ and $x > 1$

$\Rightarrow$ • f is increasing for $-2 < x < 0$ and $x > 1$ and decreasing for $x < -2$, $0 < x < 1$

• f has local maximum at 0 and minima at -2 and 1.

b) $y = \tan^{-1}(x^2)$

$y' = \frac{1}{1 + (x^2)^2} \cdot 2x = \frac{2x}{1 + x^4}$

$y' = 0 \iff x = 0$

y' is negative for $x < 0$ and positive for $x > 0$

$\Rightarrow f$ has local minimum at $x = 0$, is increasing for $x > 0$, decreasing for $x < 0$

c) $f(x) = \frac{x}{(x-1)^2} \Rightarrow f'(x) = \frac{(x-1)^2 - 2(x-1)x}{(x-1)^4} = \frac{x-3}{(x-1)^3}$?

$\rightsquigarrow f'(x)$ has crit pts at $1, 3$., $f'(x) > 0$ for $x < 1$, $x > 3$, $f'(x) < 0$ for $1 < x < 3$

· f has local maximum at $x=1$, local minimum at $x=3$
· f is increasing for $x<1$, $x>3$, decreasing for $1<x<3$

d) $f(x) = (x^2-x)^{1/3} \Rightarrow f'(x) = \frac{1}{3}(x^2-x)^{-2/3} \cdot (2x-1) = \frac{2x-1}{3\sqrt[3]{(x^2-x)^2}}$

$\rightarrow$ by 3 d) crit values at $0, \frac{1}{2}, 1$; $f'(x)>0$ for $x>\frac{1}{2}$, $f'(x)<0$ for $x<\frac{1}{2}$

$\Rightarrow$ $f(x)$ increasing for $x>1/2$, decreasing for $x<1/2$ and $f(x)$ has local local minimum at $x=1/2$

e) $f(x) = x \cdot \sin(x) + \cos(x)$ on $[0, 2\pi]$

$\rightarrow f'(x) = \sin(x) + x\cdot\cos(x) - \sin(x) = x \cdot \cos(x)$

$f'(x) = 0 \Leftrightarrow x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, $f'(x) < 0$ for $x \in (\frac{\pi}{2}, \frac{3}{2}\pi)$ and

$f'(x) > 0$ for $x \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$

$\Rightarrow$ $f(x)$ has a local maximum at $x = \frac{3}{2}\pi$, a local minimum at $x = \frac{\pi}{2}$,
$f(x)$ is increasing ~~for~~ for $x \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$ and decreasing for $x \in (\frac{\pi}{2}, \frac{3}{2}\pi)$.

9) Let $f(x) = x^5 + 5x^4$. Determine where $f(x)$ is concave up/down and find all inflection points.

$\rightarrow f'(x) = 5x^4 + 20x^3$; $f''(x) = 20x^3 + 60x^2 = 20x^2(x+3)$

$f''(x) = 0 \Leftrightarrow x=0$ or $x=-3$

$f''(x) \geq 0$ for $x \geq -3$ and $f''(x) < 0$ for $x < -3$

$\Rightarrow$ $f(x)$ has an inflection point at -3 and is convex (concave up) ~~at~~ for $x > -3$ and concave (down) for $x < -3$

10) Let $f(-3) = 4$, $f'(-3) = 0$, $f''(-3) = 7$, $f(2) = -5$, $f'(2) = 0$, $f''(2) = -6$. Identify any local extrema of f.

$\rightarrow$ · $f'(2) = 0$, $f''(2) = -6 \Rightarrow f$ has local maximum at 2.
· $f'(-3) = 0$, $f''(-3) = 7 \Rightarrow f$ has local minimum at -3.

But since $f(-3) = 4 > -6 = f(2)$ f needs to have an additional maximum and an additional minimum in the interval $(-3, 2)$ if f is continuous.