Arc length

① $y = f(x),\ a \le x \le b$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

③ $x = g(t),\ y = f(t),$

$\alpha \le t \le \beta$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

② $x = g(y),\ c \le y \le d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Section 9.3

1. Find the length of the curve $y = 2x^{3/2}$, $0 \le x \le \frac{1}{4}$.

① $L = \int_0^{\frac{1}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = 2x^{\frac{3}{2}}$

$\frac{dy}{dx} = 3x^{\frac{1}{2}} = 3\sqrt{x}$

$L = \int_0^{\frac{1}{4}} \sqrt{1 + 9x}\, dx$

$u = 1 + 9x$ ⟨ $x = \frac{1}{4},\ u = \frac{13}{4}$; $x = 0,\ u = 1$

$du = 9\,dx$

$L = \frac{1}{9} \int_1^{\frac{13}{4}} u^{\frac{1}{2}}\, du$

$L = \frac{1}{9} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{\frac{13}{4}}$

$\frac{2}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right)$

$\frac{2}{27}\left(\frac{13^{3/2}}{8} - 1\right)$

$$\boxed{\frac{2}{27}\left(\frac{13\sqrt{13}}{8} - 1\right)}$$

2. Find the length of the curve $x = y^2 - \frac{\ln(y)}{8}$ from $y = 1$ to $y = e$.

② $L = \int_1^e \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$x = y^2 - \frac{1}{8}\ln y$

$\frac{dx}{dy} = 2y - \frac{1}{8y}$

$\left(\frac{dx}{dy}\right)^2 = 4y^2 - 2 \cdot 2y \cdot \frac{1}{8y} + \frac{1}{64y^2}$

$= 4y^2 - \frac{1}{2} + \frac{1}{64y^2}$

$L = \int_1^e \sqrt{1 + 4y^2 - \frac{1}{2} + \frac{1}{64y^2}}\, dy$

$= \int_1^e \sqrt{4y^2 + \frac{1}{2} + \frac{1}{64y^2}}\, dy$

$= \int_1^e \sqrt{\left(2y + \frac{1}{8y}\right)^2}\, dy$

$= \int_1^e \left(2y + \frac{1}{8y}\right) dy$

$= \left(y^2 + \frac{1}{8}\ln y\right)\Big|_1^e$

$= e^2 + \frac{1}{8}\ln e - \left(1 + \frac{1}{8}\ln 1\right)$

$= e^2 + \frac{1}{8} - 1 = \boxed{e^2 - \frac{7}{8}}$

3. Find the length of the parametric curve $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 2$.

$x = 3t - t^3$ $\quad$ $y = 3t^2$

$\frac{dx}{dt} = 3 - 3t^2$ $\quad$ $\frac{dy}{dt} = 6t$

(3) $L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$L = \int_0^2 \sqrt{(3 - 3t^2)^2 + (6t)^2}\, dt$$

$$= \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt$$

$$= \int_0^2 \sqrt{9t^4 + 18t^2 + 9}\, dt$$

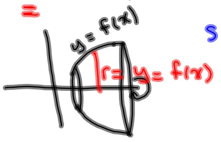
$$= \int_0^2 \sqrt{(3t^2 + 3)^2}\, dt$$

$$= \int_0^2 (3t^2 + 3)\, dt$$

$$= (t^3 + 3t)\Big|_0^2$$

$$= \boxed{14}$$

$\circlearrowleft$ $\underline{x}$-axis



$SA = \int 2\pi r \text{ arclength } \begin{matrix} dx \\ dy \\ dt \end{matrix}$

$\circlearrowleft$ y-axis

$SA = \int 2\pi r \text{ arclength } \begin{matrix} dx \\ dy \\ dt \end{matrix}$

Section 9.4

4. Find the surface area obtained by revolving the given curve about the indicated axis.

a.) $y = 2x^3$, $0 \le x \le 1$ about the x axis.

$\hookrightarrow r = y = 2x^3$

$$SA = \int_0^1 2\pi r \text{ arclength } dx \qquad \frac{dy}{dx} = 6x^2$$

$$= \int_0^1 2\pi \cdot 2x^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_0^1 x^3 \sqrt{1 + 36x^4}\, dx \qquad u = 1 + 36x^4 \begin{cases} x=1, u=37 \\ x=0, u=1 \end{cases} \quad du = 144x^3 dx$$

$$= \frac{4\pi}{144} \int_1^{37} u^{\frac{1}{2}} du$$

$$= \frac{\pi}{36} \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{37}$$

$$= \frac{\pi}{54} \left[37^{\frac{3}{2}} - 1^{\frac{3}{2}} \right] \rightarrow \boxed{\frac{\pi}{54}\left(37\sqrt{37} - 1\right)}$$

b.) $y^2 = x + 2$, $\underline{1 \le y \le 3}$ about the x axis.

$x = y^2 - 2$ $\qquad \hookrightarrow r = y$ ✓

$\frac{dx}{dy} = 2y$

$$SA = \int_1^3 2\pi r \text{ arclength } dy$$

$$= \int_1^3 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_1^3 y \sqrt{1 + 4y^2}\, dy \qquad u = 1 + 4y^2 \begin{cases} y=3, u=37 \\ y=1, u=5 \end{cases} \quad du = 8y\, dy$$

$$= \frac{2\pi}{8} \int_5^{37} u^{\frac{1}{2}} du$$

$$= \frac{\pi}{4} \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{37} \rightarrow \boxed{\frac{\pi}{6}\left(37\sqrt{37} - 5\sqrt{5}\right)}$$

c.) $y = x^2 + 1$, $0 \le x \le 1$, about the y axis.

$\hookrightarrow r = x$

$\frac{dy}{dx} = 2x$

$$SA = \int_0^1 2\pi r \,\text{arclength}\, dx$$

$$= \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$u = 1 + 4x^2$; $x = 1, u = 5$; $x = 0, u = 1$

$du = 8x\,dx$

$$= 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx$$

$$= \frac{\pi}{4}\int_1^5 u^{\frac{1}{2}}\, du = \frac{\pi}{4}\,\frac{2}{3} u^{\frac{3}{2}}\Big|_1^5 = \boxed{\frac{\pi}{6}\left(5\sqrt{5} - 1\right)}$$

d.) $y = \sqrt{4x}$, $0 \le x \le 1$, about the x axis.

$\hookrightarrow r = y$

$y^2 = 4x$ $\quad 0 \le y \le 2$

$x = \frac{1}{4}y^2$

$\frac{dx}{dy} = \frac{1}{2}y$

$$SA = \int_0^2 2\pi r \,\text{arclength}\, dy$$

$$= \int_0^2 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$u = 1 + \frac{1}{4}y^2$; $y = 2, u = 2$; $y = 0, u = 1$

$du = \frac{1}{2}y\,dy$

$$= 2\pi \int_0^2 y\sqrt{1 + \frac{1}{4}y^2}\, dy$$

$$= 4\pi \int_1^2 u^{\frac{1}{2}}\, du$$

$$= 4\pi\,\frac{2}{3} u^{\frac{3}{2}}\Big|_1^2$$

$$= \boxed{\frac{8\pi}{3}\left(2\sqrt{2} - 1\right)}$$

e.) $x = \ln(3y+1)$, $0 \le y \le 2$, about the y axis, then the x axis. Set up the integral that gives the surface area. Do not integrate.

@ y-axis: $r = x = \ln(3y+1)$

$$\frac{dx}{dy} = \frac{3}{3y+1}$$

$$SA = \int_0^2 2\pi \ln(3y+1)\sqrt{1+\left(\frac{3}{3y+1}\right)^2}\,dy$$

@ x-axis: $r = y$ ✓

$$SA = \int_0^2 2\pi y\sqrt{1+\left(\frac{3}{3y+1}\right)^2}\,dy$$

f.) $x = \sin(3t)$, $y = \cos(3t)$, $0 \le t \le \frac{\pi}{12}$. about the y axis.

↳ $r = x = \sin(3t)$

$$SA = \int_0^{\frac{\pi}{12}} 2\pi \sin(3t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$x = \sin(3t)$ $\qquad$ $y = \cos(3t)$

$\frac{dx}{dt} = 3\cos(3t)$ $\qquad$ $\frac{dy}{dt} = -3\sin(3t)$

$$SA = \int_0^{\frac{\pi}{12}} 2\pi \sin(3t)\sqrt{\underbrace{9\cos^2(3t)+9\sin^2(3t)}_{\substack{9(\cos^2(3t)+\sin^2(3t)) \\ =9}}}\,dt$$

$$= \int_0^{\frac{\pi}{12}} 2\pi \sin(3t)\sqrt{9}\,dt$$

$$= 6\pi\int_0^{\frac{\pi}{12}} \sin(3t)\,dt$$

$$= 6\pi\left[-\frac{1}{3}\cos(3t)\right]\Big|_0^{\frac{\pi}{12}}$$

$$= -2\pi\left[\cos\frac{\pi}{4} - \cos(0)\right)$$

$$= \boxed{-2\pi\left(\frac{\sqrt{2}}{2}-1\right)}$$

Section 10.1

5. Find the fourth term of the sequence $\{\frac{n}{n+1}\}_{n=2}^{\infty}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{3}{4}$

$a_4 = \frac{4}{5}$

$\boxed{a_5 = \frac{5}{6}}$

6. Find a general formula for the sequence

$\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots = \frac{1}{2\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{2\cdot 4}, \frac{1}{2\cdot 5}$

$= \left\{ \frac{1}{2n} \right\}_{n=2}^{\infty} = \left\{ \frac{1}{2(n+1)} \right\}_{n=1}^{\infty}$

7. Find a general formula for the sequence

$-\frac{1}{3}, \frac{1}{7}, -\frac{1}{11}, \frac{1}{15}, \ldots$

alternating signs $\rightarrow (-1)^n$ or $(-1)^{n+1}$

odd numbers: $2n+1$ ✗

$2n+3$ ✗

$4n-1$ ✓

$\left\{ \frac{(-1)^n}{4n-1} \right\}_{n=1}^{\infty}$

8. Determine whether the following sequences converge or diverge. If the sequence converges, find the limit. If the sequence diverges, explain why.

a.) $a_n = \dfrac{n^3}{n^2 + 500n - 2}$

if $\lim_{n\to\infty} a_n$ exists, $\{a_n\}$ converges

if $\lim_{n\to\infty} a_n$ dne, $\{a_n\}$ diverges.

$$\lim_{n\to\infty} \frac{n^3}{n^2+500n-2} = \infty$$ [higher degree on top]

$\boxed{\text{diverges}}$

b.) $a_n = \ln(2n+1) - \ln(5n+4) = \ln\left(\dfrac{2n+1}{5n+4}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{2n+1}{5n+4}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+1}{5n+4}\right)$$

$$= \boxed{\ln \frac{2}{5}}$$ $\boxed{\text{converges}}$

c.) $a_n = \dfrac{5\cos n}{n}$

$-1 \le \cos n \le 1$

$-5 \le 5\cos n \le 5$

$$-\frac{5}{n} \le \frac{5\cos n}{n} \le \frac{5}{n}$$

$-\frac{5}{n} \to 0$ as $n\to\infty$, $\frac{5}{n} \to 0$ as $n\to\infty$

$\hookrightarrow \lim_{n\to\infty} \dfrac{5\cos n}{n} = 0$ by squeeze theorem.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$

d.) $a_n = \dfrac{(-1)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n}$$

$$= 0 \quad \therefore \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

converges

e.) $a_n = \dfrac{(-1)^n n}{5n+6}$

$$\lim_{n\to\infty} \left| \frac{(-1)^n n}{5n+6} \right| = \lim_{n\to\infty} \frac{n}{5n+6} = \frac{1}{5}$$

n even $\lim_{n\to\infty} \dfrac{(-1)^n n}{5n+6} \to \dfrac{1}{5}$

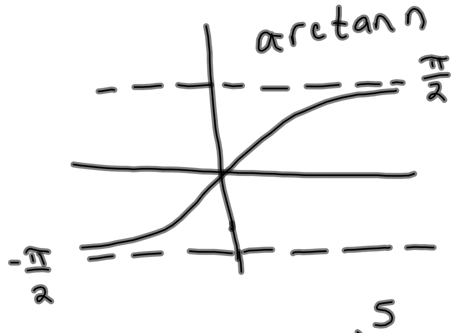
n odd $\lim_{n\to\infty} \dfrac{(-1)^n n}{5n+6} \to -\dfrac{1}{5}$

since limits are unique,

$\lim_{n\to\infty} \dfrac{(-1)^n n}{5n+6}$ d n e

diverges by oscillation.

f.) $a_n = \dfrac{(\arctan n)^5}{n^2}$



$$\lim_{n\to\infty} \frac{(\arctan n)^5}{n^2} \to \frac{\left(\frac{\pi}{2}\right)^5}{\infty} = \boxed{0}$$

g.) $a_n = \sqrt{n^2 + 4n} - n$ $= \infty - \infty = 0$ $\boxed{\text{NO!}}$

$$\lim_{n\to\infty} \frac{\left(\sqrt{n^2+4n} - n\right)\left(\sqrt{n^2+4n} + n\right)}{\sqrt{n^2+4n} + n}$$

$$\lim_{n\to\infty} \frac{\cancel{n^2} + 4n - \cancel{n^2}}{\sqrt{n^2+4n} + n}$$

$$\lim_{n\to\infty} \frac{4n}{\sqrt{n^2+4n} + n} = \lim_{n\to\infty} \frac{4\cancel{n}}{2\cancel{n}}$$

$$= \boxed{2}$$

9. Prove the sequence $a_n = \frac{\ln n}{n}$ is a decreasing sequence.

recall: $f(x)$ decreases if $f'(x) < 0$

$$\frac{d}{dn}\left(\frac{\ln n}{n}\right) = \frac{\frac{1}{n}(n) - \ln n(1)}{n^2}$$

$$= \frac{1 - \ln n}{n^2} \leftarrow \text{eventually negative}$$

10. For the recursive sequence given, find the 3rd term and find the value of the limit.

$a_1 = 2$, $a_{n+1} = 2 + \frac{1}{4}a_n$.

$\boxed{a_{n+1} = f(a_n)}$

$\boxed{a_1 = 2}$, $a_{n+1} = 2 + \frac{1}{4}a_n$

$a_1 = 2$, $a_2 = 2 + \frac{1}{4}a_1$

$$\boxed{a_2 = 2 + \frac{1}{4}(2) = 2 + \frac{1}{2} = \frac{5}{2}}$$

$$\boxed{a_3 = 2 + \frac{1}{4}a_2 = 2 + \frac{1}{4}\cdot\frac{5}{2} = 2 + \frac{5}{8} = \frac{21}{8}}$$

Find $\lim_{n\to\infty} a_n$:

$\lim_{n\to\infty} a_n = L$

$$a_{n+1} = 2 + \frac{1}{4}a_n$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(2 + \frac{1}{4}a_n\right)$$

$$L = 2 + \frac{1}{4}L$$

$$\frac{3}{4}L = 2$$

$$\boxed{L = \frac{8}{3}}$$