Section 8.3

$a^2 - x^2 \longrightarrow x = a\sin\theta$

$x^2 - a^2 \longrightarrow x = a\sec\theta$

$x^2 + a^2 \longrightarrow x = a\tan\theta$

1. $\displaystyle\int \frac{2}{x\sqrt{x^2-4}}\,dx$

$x = 2\sec\theta \longrightarrow \frac{x}{2} = \sec\theta \to \sec^{-1}\left(\frac{x}{2}\right) = \theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{2}{\cancel{2\sec\theta}\sqrt{\underbrace{4\sec^2\theta - 4}_{\substack{4(\sec^2\theta-1)\\ 4\tan^2\theta}}}} \cdot \cancel{2\sec\theta}\tan\theta\,d\theta$$

$$\int \frac{\cancel{2\tan\theta}\,d\theta}{\cancel{2\tan\theta}} = \int d\theta$$

$= \theta + C$

$= \boxed{\sec^{-1}\left(\frac{x}{2}\right) + C}$

2. $\displaystyle\int_0^{1/3} \frac{3}{\sqrt{1+9y^2}}\,dy = \int_0^{\frac{1}{3}} \frac{3}{\sqrt{1+(3y)^2}}\,dy$

$3y = \tan\theta$

$y = \frac{1}{3} \to 1 = \tan\theta \to \theta = \frac{\pi}{4}$

$y = 0 \to 0 = \tan\theta \to \theta = 0$

$3\,dy = \sec^2\theta\,d\theta$

$dy = \frac{1}{3}\sec^2\theta$

$$\int_0^{\frac{\pi}{4}} \frac{\cancel{3}}{\sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}}}\ \frac{1}{\cancel{3}}\sec^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\,d\theta}{\sec\theta}$$

$$* \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta \ *$$

$$\ln\left|\sec\theta + \tan\theta\right| \Big|_0^{\frac{\pi}{4}}$$

$$\ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$\ln\left|\sqrt{2}+1\right| - \cancel{\ln|1+0|} \quad \ln 1 = 0$$

$$\boxed{\ln\left(\sqrt{2}+1\right)}$$

3. $\int \frac{dx}{\sqrt{x^2+4x+8}}$

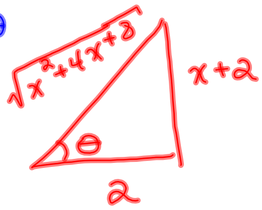
$\underbrace{x^2+4x+4}+8-4$ $\qquad \left(\frac{4}{2}\right)^2=4$

$(x+2)^2+4$

$\int \frac{dx}{\sqrt{(x+2)^2+4}}$

$x+2=2\tan\theta \rightarrow \frac{x+2}{2}=\tan\theta$

$dx=2\sec^2\theta\, d\theta$



$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{\underbrace{4\tan^2\theta+4}_{4(\tan^2\theta+1)}}}=\int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\sec\theta}$

$4\sec^2\theta$

$=\int \sec\theta\, d\theta$

$=\ln|\sec\theta+\tan\theta|+C$

$=\boxed{\ln\left|\frac{\sqrt{x^2+4x+8}}{2}+\frac{x+2}{2}\right|+C}$

4. $\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-(2x)^2}\,dx$

$2x=\sin\theta$

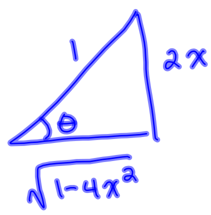
$2\,dx=\cos\theta\, d\theta$

$=\int \sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}\cdot\frac{1}{2}\cos\theta\, d\theta$ $\qquad dx=\frac{1}{2}\cos\theta\, d\theta$

$=\frac{1}{2}\int \cos^2\theta\, d\theta$

$\frac{o}{h}\quad \frac{2x}{1}=\sin\theta$

$=\frac{1}{2}\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$



$2\sin\theta\cos\theta$

$=\frac{1}{4}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C$

$=\frac{1}{4}(\theta+\sin\theta\cos\theta)$

$=\boxed{\frac{1}{4}\left(\sin^{-1}(2x)+(2x)\left(\sqrt{1-4x^2}\right)\right)+C}$

Section 8.4

5. $\int \frac{x+2}{x^2-2x-15}\,dx$

① is degree in denominator > degree in numerator?

$\boxed{\text{yes}}$

② Factor denominator

$$\int \frac{x+2}{(x-5)(x+3)}\,dx$$

PFD : $\frac{x+2}{(x-5)(x+3)} = \left(\frac{A}{x-5} + \frac{B}{x+3}\right)(x-5)(x+3)$

$x+2 = A(x+3) + B(x-5)$

$x=5:\ 7 = A(8) \to \boxed{A = \frac{7}{8}}$

$x=-3:\ -1 = B(-8) \to \boxed{B = \frac{1}{8}}$

Section 8.4

5. $\int \frac{x+2}{x^2-2x-15}\,dx = \int \left(\frac{\frac{7}{8}}{x-5} + \frac{\frac{1}{8}}{x+3}\right) dx$

$u = x-5$, $du = dx$, $\int \frac{du}{u} = \ln|u|$

$u = x+3$, $du = dx$, $\int \frac{du}{u} = \ln|u|$

$$= \boxed{\frac{7}{8}\ln|x-5| + \frac{1}{8}\ln|x+3| + C}$$

6. $\int \frac{x^4+x-4}{x^2-1}\,dx$

① degree <u>not</u> higher in denominator

LONG DIVISION

$$\begin{array}{r} x^2 + 1 \leftarrow Q \\ D \rightarrow x^2-1 \,\big)\, x^4 + x - 4 \\ -x^4 + x^2 \\ \hline x^2 + x - 4 \\ -x^2 + 1 \\ \hline x - 3 \leftarrow R \end{array}$$

$Q + \frac{R}{D}$

$x^2 + 1 + \frac{x-3}{x^2-1}$

$\frac{x-3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

6. $\int \frac{x^4+x-4}{x^2-1}\,dx = \int \left(x^2+1+\frac{x-3}{x^2-1}\right) dx$

$x-3 = A(x-1) + B(x+1)$

$x=1:\ -2 = B(2) \to \boxed{B=-1}$

$x=-1:\ -4 = A(-2) \to \boxed{A=2}$

$$= \int \left(x^2+1+\frac{2}{x+1} - \frac{1}{x-1}\right) dx$$

$$= \boxed{\frac{x^3}{3} + x + 2\ln|x+1| - \ln|x-1| + C}$$

7. $\int_1^2 \frac{dx}{x(x^2+2x+1)}$

① higher degree on bottom ✓

② Factor denominator

$$\int_1^2 \frac{dx}{x(x+1)^2}$$

PFD: $\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$1 = A(x+1)^2 + Bx(x+1) + Cx$

$$= \int_1^2 \left(\frac{1}{x} + \frac{-1}{x+1} - \frac{1}{(x+1)^2}\right) dx$$

$u = x+1$

$du = dx$

$\int \frac{du}{u^2} = -\frac{1}{u}$

$= -\frac{1}{x+1}$

$x=-1: \quad 1 = C(-1) \to \boxed{C=-1}$

$x=0: \quad 1 = A(1) \to \boxed{A=1}$

$x=1: \quad 1 = 1(4) + B(2) - 1$

$1 = 3 + 2B$

$-2 = 2B \to \boxed{B=-1}$

$$= \left(\ln|x| - \ln|x+1| + \frac{1}{x+1}\right)\Big|_1^2$$

$$= \boxed{\ln 2 - \ln 3 + \frac{1}{3} - \left(\ln 1 - \ln 2 + \frac{1}{2}\right)}$$

$x^2+1 =$ irreducible quadratic

8. $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)} dx$

PFD: $\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$3x^2-4x+5 = A(x^2+1) + (Bx+C)(x-1)$

$x=1: \quad 4 = A(2) \to \boxed{A=2}$

$3x^2-4x+5 = 2x^2+2+Bx^2-Bx+Cx-C$

$3x^2-4x+5 = (2+B)x^2 + (-B+C)x + 2-C$

$3 = 2+B \to \boxed{B=1}$

$-4 = -B+C \to -4 = -1+C \to \boxed{C=-3}$

check: $5 = 2-C \to 5 = 2-(-3)$ ✓

8. $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)} dx = \int \left(\frac{2}{x-1} + \frac{x-3}{x^2+1}\right) dx$

split!!

$$= \int \left(\frac{2}{x-1} + \frac{x}{x^2+1} - \frac{3}{x^2+1}\right) dx$$

$\int \frac{dx}{x^2+1} = \arctan x$

u-sub

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln(x^2+1)$

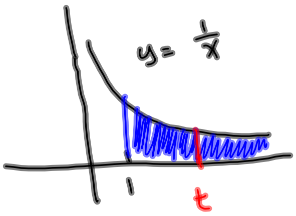
$$= \boxed{2\ln|x-1| + \frac{1}{2}\ln(x^2+1) - 3\arctan x + C}$$

Section 8.9

9. Determine whether the following improper integrals converge or diverge. If it converges, find the value of the integral. If it diverges, explain why.

a.) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

finite value = converge
infinite value = diverge



$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x}$$

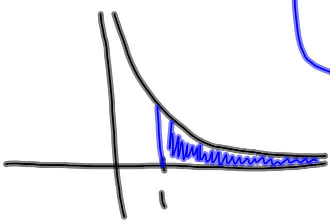
$$= \lim_{t\to\infty} \ln x \Big|_1^t$$

$$= \lim_{t\to\infty} (\overset{\infty}{\cancel{\ln t}} - \ln 1)$$

$$\boxed{= \infty}$$

integral diverges

b.) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$



$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\overset{0}{\cancel{-\frac{1}{t}}} + 1\right)$$

$$= \boxed{1}$$

integral converges

c.) $\int_e^\infty \frac{1}{x(\ln x)^4}\,dx = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^4}$

$u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{1}{u^4}\,du = -\frac{1}{3u^3}$

$= -\frac{1}{3(\ln x)^3}$

$= \lim_{t\to\infty} \left. -\frac{1}{3(\ln x)^3} \right|_e^t$

$= \lim_{t\to\infty} \left(\underbrace{-\frac{1}{3(\ln t)^3}}_{\to 0} + \frac{1}{3(\ln e)^3} \right)$

$= \boxed{\frac{1}{3} \quad \text{integral converges}}$

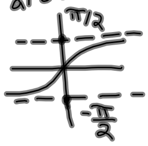
d.) $\int_{-\infty}^\infty \frac{dx}{x^2+9}$ split in two

A side:

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

① $\int_{-\infty}^0 \frac{dx}{x^2+9} +$ ② $\int_0^\infty \frac{dx}{x^2+9}$

arctanx



① : $\lim_{t\to -\infty} \int_t^0 \frac{dx}{x^2+9} = \lim_{t\to-\infty} \left. \frac{1}{3}\arctan\frac{x}{3} \right|_t^0$

$= \lim_{t\to-\infty} \frac{1}{3}\left[\underbrace{\arctan 0}_{\to 0} - \underbrace{\arctan \frac{t}{3}}_{\to \arctan(-\infty) = -\frac{\pi}{2}} \right]$

$= \boxed{\frac{\pi}{6}}$ ← converges

evaluate ②

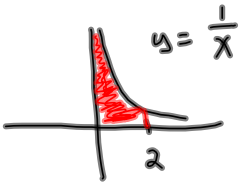
② : $\lim_{t\to\infty} \int_0^t \frac{dx}{x^2+9} = \lim_{t\to\infty} \left. \frac{1}{3}\arctan\frac{x}{3} \right|_0^t$

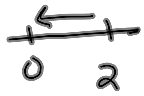
$= \lim_{t\to\infty} \frac{1}{3}\left[\underbrace{\arctan\frac{t}{3}}_{\to \frac{\pi}{2}} - \underbrace{\arctan 0}_{\to 0} \right]$

$\boxed{= \frac{\pi}{6}}$

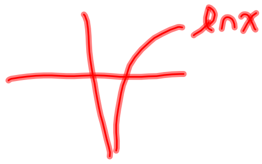
$\int_{-\infty}^\infty \frac{dx}{x^2+9} = \frac{\pi}{6} + \frac{\pi}{6} = \boxed{\frac{\pi}{3}}$

e.) $\int_{\boxed{0}}^{2} \frac{1}{x}\,dx$





$$\lim_{t\to 0^+} \int_t^2 \frac{dx}{x} = \lim_{t\to 0^+} \ln|x| \Big|_t^2$$

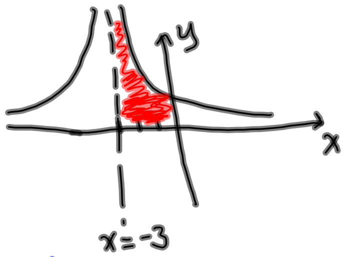


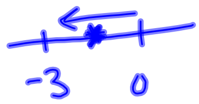
$$= \lim_{t\to 0^+} (\ln 2 - \ln t) \qquad \to -\infty$$

$$= \ln 2 + \infty$$

$$= \boxed{\infty \quad \text{diverges}}$$

f.) $\int_{-3}^{0} \frac{dx}{(x+3)^2}$





$$\lim_{t\to -3^+} \int_t^0 \frac{dx}{(x+3)^2} = \lim_{t\to -3^+} \left(\frac{-1}{x+3}\right)\Big|_t^0 \qquad \frac{1}{0^+} = +\infty$$

$u = x+3$

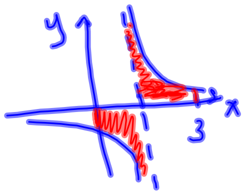
$du = dx$

$\int \frac{1}{u^2} = -\frac{1}{u} = \frac{-1}{x+3}$

$$\lim_{t\to -3^+} \left(-\frac{1}{3} + \frac{1}{t+3}\right)$$

$$= \boxed{\infty} \quad \boxed{\text{diverges}}$$

g.) $\displaystyle\int_0^3 \frac{1}{2x-1}\,dx = \int_0^3 \frac{1}{2(x-\frac{1}{2})}\,dx$



split integral at $x=\frac{1}{2}$!

$$\textcircled{1}\ \int_0^{\frac{1}{2}} \frac{dx}{2(x-\frac{1}{2})} + \textcircled{2}\ \int_{\frac{1}{2}}^3 \frac{dx}{2(x-\frac{1}{2})}$$

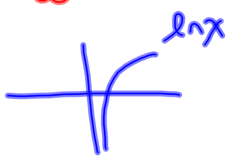
$$\textcircled{2}\quad \lim_{t\to\frac{1}{2}^+} \int_t^3 \frac{dx}{2(x-\frac{1}{2})} = \lim_{t\to\frac{1}{2}^+} \frac{1}{2}\ln\left|x-\frac{1}{2}\right| \Big|_t^3$$

$$= \lim_{t\to\frac{1}{2}^+} \frac{1}{2}\left[\ln\frac{5}{2} - \ln\left|t-\frac{1}{2}\right|\right]$$

$\ln(\text{"}0\text{"})$
$= -\infty$

$$= \frac{1}{2}\left[\ln\frac{5}{2}+\infty\right]$$

$= \infty$ integral diverges



$$\boxed{\therefore \int_0^3 \frac{dx}{2(x-\frac{1}{2})} \text{ diverges}}$$

10. Determine whether the following integrals converge or diverge using the comparison theorem:

a.) $\int_1^\infty \frac{1}{x+e^{5x}}\,dx \leq \int_1^\infty \frac{1}{e^{5x}}\,dx$

$= \int_1^\infty e^{-5x}\,dx$

$= \lim_{t\to\infty} \int_1^t e^{-5x}\,dx$

$= \lim_{t\to\infty} -\frac{1}{5}e^{-5x}\Big|_1^t$

$= \lim_{t\to\infty} -\frac{1}{5}\left[e^{-5t} - e^{-5}\right]$

$e^{-\infty} = 0$

$= \frac{1}{5e^5}$ → larger converges, so does smaller.

comparison theorem:

if $0 \leq f(x) \leq g(x)$ on $[a,\infty)$

① $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

"smaller diverges, so must larger"

② $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges

"larger converges, so must smaller"

b.) $\int_2^\infty \frac{x+1}{x^{3/2}-x}\,dx \geq \int_2^\infty \frac{x}{x^{3/2}}\,dx = \int_2^\infty x^{-\frac{1}{2}}\,dx$

$= \lim_{t\to\infty} \int_2^t x^{-\frac{1}{2}}\,dx$

$= \lim_{t\to\infty} 2\sqrt{x}\Big|_2^t$

$= \lim_{t\to\infty} \left(2\sqrt{t} - 2\sqrt{2}\right) = \boxed{\infty}$

∴ smaller diverges so must larger

c.) $\int_1^\infty \frac{\cos^2 x}{x^4}\,dx \leq \int_1^\infty \frac{1}{x^4}\,dx$

$\cos x \leq 1$

$\cos^2 x \leq 1$

$\lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} -\frac{1}{3x^3}\Big|_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \boxed{\frac{1}{3}}$

larger converges so does smaller