Section 6.5

1. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $2du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= \int e^u\, 2\,du$

$= 2\int e^u\,du$

$= 2e^u + C \to \boxed{2e^{\sqrt{x}} + C}$

2. $\int x \sin(\underbrace{x^2 - 2}_{u})\,dx$ $\frac{du}{2}$

$u = x^2 - 2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2}\int \sin u\,du$

$= -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos(x^2 - 2) + C$

3. $\int_0^1 \frac{6x+1}{x^2+1}\,dx$

Recall: $\int \frac{dx}{x^2+1} = \arctan x + C$

$\int_0^1 \frac{6x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx$

$\to \arctan x \Big|_0^1 = \arctan(1) - \cancel{\arctan 0}^{\,0}$

$= \frac{\pi}{4}$

$u = x^2 + 1$

$du = 2x\,dx$

$\int \frac{6x}{x^2+1}\,dx = 3\int \frac{du}{u}$

$= 3\ln|u|$

$= 3\ln(x^2+1)$

$\boxed{\begin{aligned}\text{Answer: } & 3\ln 2 + \frac{\pi}{4} \\ & = \ln 8 + \frac{\pi}{4}\end{aligned}}$

$\to 3\ln(x^2+1)\Big|_0^1 = 3\ln 2 - 3\cancel{\ln(1)}^{\,0}$

$= 3\ln 2$

4. $\int \frac{x}{(\underbrace{x+1}_{u})^3}\,dx$ $x \to u-1$, $dx \to du$

$\to u = x + 1$

$du = dx$

$\int \frac{u-1}{u^3}\,du = \int\left(\frac{u}{u^3} - \frac{1}{u^3}\right)du$

$= \int(u^{-2} - u^{-3})\,du$

$= \frac{u^{-1}}{-1} - \frac{u^{-2}}{-2} + C$

$= -\frac{1}{u} + \frac{1}{2u^2} + C$

$= \boxed{-\frac{1}{x+1} + \frac{1}{2(x+1)^2} + C}$

Section 7.1

5. Find the area of the region bounded by the following pairs of curves.

a.) $y = x + 2$, $y = x^2$

First, sketch the bounded area.

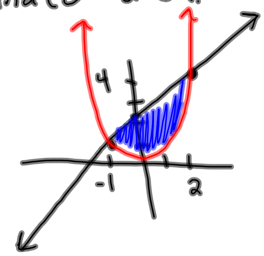
$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \quad y = 4$

$x = -1, \quad y = 1$



$$A = \int_{-1}^{2} \left(x + 2 - x^2\right) dx$$

$$A = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2}$$

$$A = \underline{\frac{9}{2}}$$

b.) $x + y^2 = 2$, $x + y = 0$

$x = 2 - y^2, \quad x = -y$

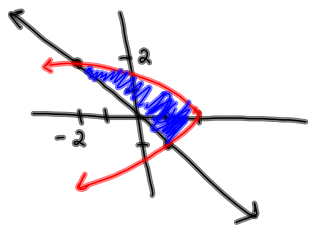
$2 - y^2 = -y$

$0 = y^2 - y - 2$

$0 = (y-2)(y+1)$

$y = 2, \quad x = -2$

$y = -1, \quad x = 1$

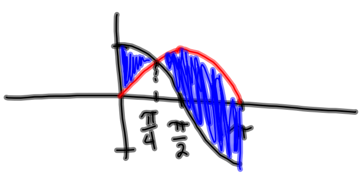


$$A = \int_{-1}^{2} \left(2 - y^2 - (-y)\right) dy$$

$$A = \left(2y - \frac{y^3}{3} + \frac{y^2}{2}\right)\Big|_{-1}^{2}$$

$$\boxed{A = \frac{9}{2}}$$

c.) $y = \cos x$, $y = \sin x$, $x = 0$, $x = \pi$



$$A = \int_{0}^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx$$

$$A = (\sin x + \cos x)\Big|_{0}^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\pi}$$

$$A = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) + \left(1 - 0 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right)$$

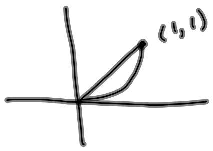
$$A = \sqrt{2} - 1 + 1 + \sqrt{2}$$

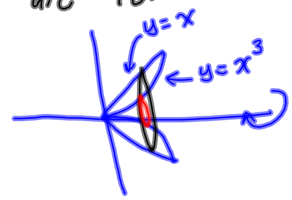
$$\boxed{A = 2\sqrt{2}}$$

Section 7.2 and 7.3

6. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^3$ (first quadrant only) about the x-axis.

① sketch region you are rotating.





$R = x$

$r = x^3$

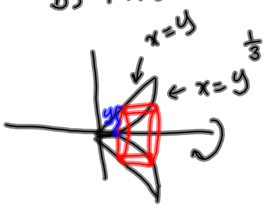
a) Find volume using washers

$$V = \pi \int_0^1 (x^2 - x^6)\, dx$$

$$V = \pi \left(\frac{x^3}{3} - \frac{x^7}{7} \right) \Big|_0^1$$

$$\boxed{V = \pi \left(\frac{4}{21} \right)}$$

b) Find volume using shells

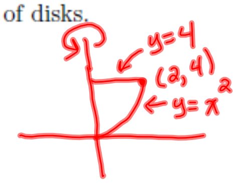


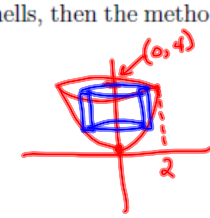
$r = y$

$h = y^{\frac{1}{3}} - y$

$$V = \int_0^1 2\pi \underbrace{y}_{r} \underbrace{(y^{\frac{1}{3}} - y)}_{h}\, dy$$

7. Find the volume of the solid obtained by rotating the region in the first quadrant bounded by $y = x^2$, $y = 4$, and $x = 0$ about the y axis by first using the method of shells, then the method of disks.





shells: $r = x$

$h = 4 - x^2$

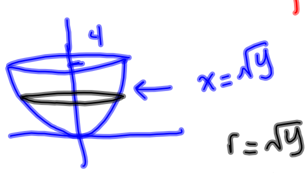
$$V = \int_0^2 2\pi x (4 - x^2)\, dx$$

$$= 2\pi \int_0^2 (4x - x^3)\, dx$$

$$= 2\pi \left(2x^2 - \frac{x^4}{4} \right) \Big|_0^2$$

$$= 2\pi (8 - 4) = \boxed{8\pi}$$

disk:



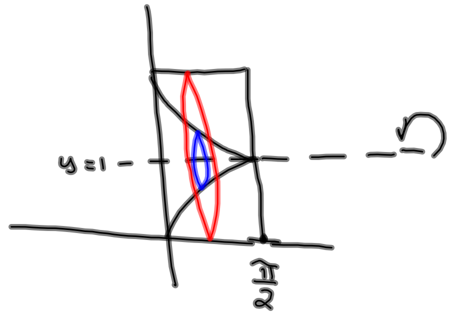
$r = \sqrt{y}$

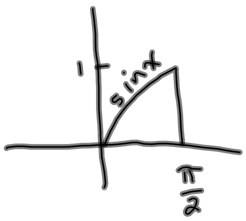
$$V = \pi \int_0^4 (\sqrt{y})^2\, dy$$

$$= \pi \int_0^4 y\, dy$$

$$= \boxed{8\pi}$$

8. Let R be the region bounded by $y = \sin x$, $x = 0$, $x = \frac{\pi}{2}$ and $y = 0$. Using the method of washers, set up the integral that gives the volume of the solid obtained by rotating R about the line $y = 1$, Do not evaluate the integral.



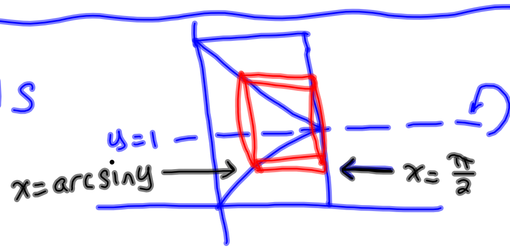


$R = 1$

$r = 1 - \sin x$

$$V = \int_0^{\frac{\pi}{2}} \pi \left[(1)^2 - (1 - \sin x)^2 \right] dx$$

using shells

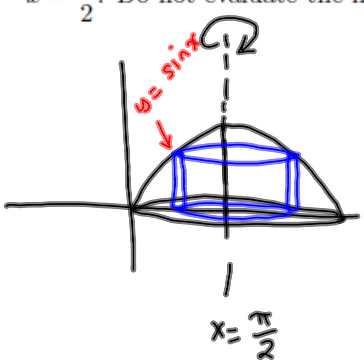


$r = 1 - y$

$h = \frac{\pi}{2} - \arcsin y$

$$V = \int_0^1 2\pi (1-y)\left(\frac{\pi}{2} - \arcsin y\right) dy$$

9. Let R be the region bounded by $y = \sin x$, $x = 0$, $x = \frac{\pi}{2}$ and $y = 0$. Using the method of cylindrical shells, set up the integral that gives the volume of the solid obtained by rotating R about the line $x = \frac{\pi}{2}$. Do not evaluate the integral.

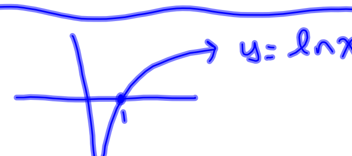


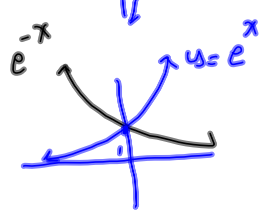
$r = \frac{\pi}{2} - x$

$h = \sin x$

$$V = \int_0^{\frac{\pi}{2}} 2\pi \left(\frac{\pi}{2} - x\right) \sin x \, dx$$

other parent functions:

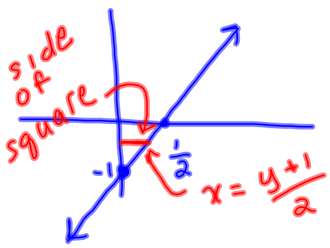




10. Find the volume of the solid whose base is the region bounded by the line $y = 2x - 1$, $x = 0$ and $y = 0$. Cross sections perpendicular to the y-axis are squares. $\rightarrow dy$

① draw base of solid

② variable of integration = axis cross sections are perpendicular to.



③ $V = \int_{-1}^{0} (A_{square})\, dy$

$A_{square} = (\text{side})^2, \quad \text{side} = \frac{y+1}{2}$

$= \left(\frac{y+1}{2}\right)^2$

$V = \int_{-1}^{0} \left(\frac{y+1}{2}\right)^2 dy$

$= \frac{1}{4} \int_{-1}^{0} (y^2 + 2y + 1)\, dy$

Section 7.4

11. The force required to stretch a spring from a natural length of 1 foot to a length of 1.5 feet is 25 pounds. How much work in foot pounds is done in stretching the spring from 1.25 to 1.5 feet?

Hooke's Law: $\underbrace{f(x)}_{\text{force}} = kx$ ← units beyond natural length.

$25 \text{ lbs} = k\left(\frac{1}{2}\text{ foot}\right)$

$25 = \frac{1}{2}k$

$\boxed{k = 50}$

$f(x) = 50x$

work: $1.25 = 0.25$ beyond natural

to $1.5 = 0.5$ feet beyond natural

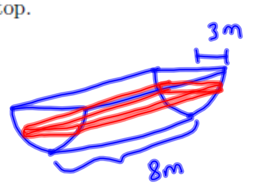
$$W = \int_{\frac{1}{4}}^{\frac{1}{2}} 50x\,dx$$

$$= 25x^2 \Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

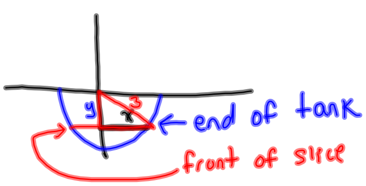
$$= \boxed{25\left(\frac{1}{4} - \frac{1}{16}\right) \text{ ft-lbs}}$$

12. A tank contains water and has the shape described below. Find the work required to pump all of the water out of the tank. Assume that $\rho = 1000$ is the density of water (in kg/m^3) and $g = 9.8$ is the acceleration due to gravity (in m/s^2).

a.) The tank is a trough 8 m long. The end of the trough is a semi circle with radius 3 m, diameter at the top.



$\boxed{\rho g = 9800 \frac{N}{m^3}}$



① $V_s = (2x)(8)(dy)$

$x^2 + y^2 = 9$

$x = \sqrt{9 - y^2}$

$V_s = \left(2\sqrt{9-y^2}\right)(8)\,dy$

$V_s = 16\sqrt{9-y^2}\,dy$

② $F_s = \rho g V_s = (9800)16\sqrt{9-y^2}\,dy$

③ $W_s = (F_s)(d_s)$ ← distance slice moves to reach top

$d_s = y$

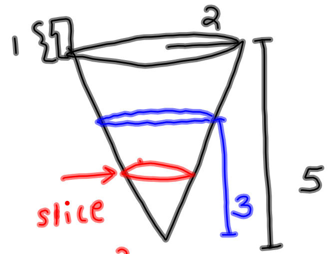
$W_s = \left[(9800)(16)\sqrt{9-y^2}\,dy\right] y$

$$W = \int_0^3 (9800)(16)\,y\sqrt{9-y^2}\,dy$$

u-sub

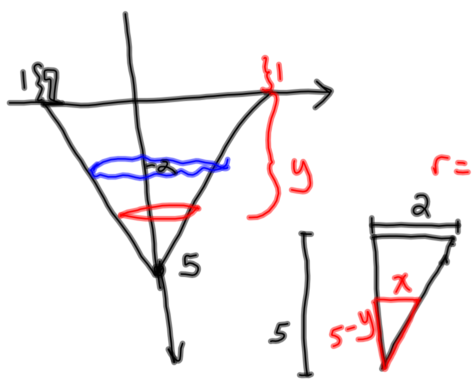
$u = 9 - y^2$

b.) The tank has the shape of an upright circular cone with height 5 m and radius 2m. In addition, there is a 1 meter high spout at the top of the cone from which the water exits the tank. If the tank is initially full to a water depth of 3 m, find the work required to pump all of the water out of the spout.



$V_s = \pi r^2 dy$

$V_s = \pi\left(\frac{4}{25}(5-y)^2\right)dy$



$r = x$

$\frac{x}{5-y} = \frac{2}{5}$

$x = \frac{2}{5}(5-y)$

$F_s = \pi \rho g \frac{4}{25}(5-y)^2 dy$

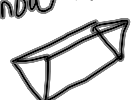
$d = y+1$

$d = y+3$

$W_s = \pi \rho g \cdot \frac{4}{25}(5-y)^2(y+3)\,dy$

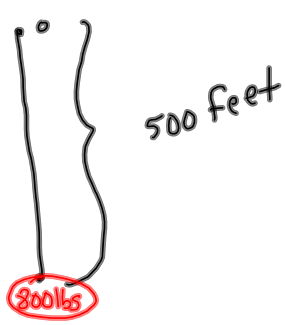
$W = \int_2^5 4 \rho g \frac{\pi}{25}(5-y)^2(y+1)\,dy$

also know how to do



hemispheres

13. A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mineshaft 500 feet deep. Find the work done.



$W = \int_0^{500} 2x\,dx + (800\text{ lbs})(500\text{ ft})$

$= x^2\Big|_0^{500} + (800)(500)$ ft lbs

Section 7.5

14. Find the average value of $f(x) = x\sqrt{x+2}$ over the interval $[-1, 2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} x\sqrt{x+2}\,dx$$

$$= \frac{1}{3}\int_{-1}^{2} x\sqrt{x+2}\,dx$$

$u = x+2$ ⟨ $x=2, u=4$; $x=-1, u=1$

$du = dx$

$$= \frac{1}{3}\int_1^4 (u-2)\sqrt{u}\,du = \frac{1}{3}\int_1^4 \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right)\Big|_1^4$$

$$= \frac{1}{3}\left(\frac{2}{5}(32) - \frac{4}{3}(8) - \left(\frac{2}{5} - \frac{4}{3}\right)\right)$$

Section 8.1

15. $\int \sqrt{x}\ln x\,dx$

Parts: $\int u\,dv = uv - \int v\,du$

$u = \ln x \qquad dv = x^{\frac{1}{2}}\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

$$\int \sqrt{x}\ln x\,dx = uv - \int v\,du$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}\left(x^{\frac{3}{2}} \cdot \frac{1}{x}\right)dx \qquad \to x^{\frac{1}{2}}$$

$$= \boxed{\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}} + C}$$

16. $\int_0^1 \frac{x}{e^{3x}}\,dx = \int_0^1 xe^{-3x}\,dx$

$u = x \qquad dv = e^{-3x}\,dx$

$du = dx \qquad v = -\frac{1}{3}e^{-3x}$

$$= uv\Big|_0^1 - \int_0^1 v\,du$$

$$= -\frac{1}{3}xe^{-3x}\Big|_0^1 - \int_0^1 -\frac{1}{3}e^{-3x}\,dx$$

$$= -\frac{1}{3}xe^{-3x}\Big|_0^1 - \frac{1}{9}e^{-3x}\Big|_0^1$$

$$= \left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)\Big|_0^1$$

$$\boxed{= -\frac{1}{3}e^{-3} - \frac{1}{9}e^{-3} - \left(0 - \frac{1}{9}\right)}$$

17. $\int x^2 \cos(2x)\,dx$

$u = x^2 \qquad dv = \cos(2x)\,dx$

$du = 2x\,dx \qquad v = \frac{1}{2}\sin(2x)$

$$\int x^2\cos(2x)\,dx = uv - \int v\,du$$

$$= \frac{1}{2}x^2\sin(2x) - \underbrace{\int x\sin(2x)\,dx}$$

parts:

$u = x \qquad dv = \sin(2x)$

$du = dx \qquad v = -\frac{1}{2}\cos(2x)$

$$uv - \int v\,du = -\frac{1}{2}x\cos(2x) + \int \frac{1}{2}\cos(2x)\,dx$$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)$$

$$\boxed{\frac{1}{2}x^2\sin(2x) - \left(-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)\right) + C}$$

18. $\int_0^{1/2} \arcsin x \, dx$

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$$\int_0^{\frac{1}{2}} \arcsin x \, dx = uv\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} v\,du$$

u-sub

$u = 1-x^2$

$-\frac{1}{2}\int u^{-\frac{1}{2}} du$

$-\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{u}$

$= -\sqrt{1-x^2}$

$$= x\arcsin x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

$$= \left(x\arcsin x + \sqrt{1-x^2}\right)\Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{1-\frac{1}{4}} - (0+1) = \boxed{\frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1}$$

19. $\int e^{2x} \cos x \, dx$

$u = e^{2x} \qquad dv = \cos x \, dx$

$du = 2e^{2x} dx \qquad v = \sin x$

$$\int e^{2x}\cos x \, dx = e^{2x}\sin x - \underbrace{\int 2e^{2x}\sin x \, dx}$$

$u = 2e^{2x} \qquad dv = \sin x \, dx$

$du = 4e^{2x} dx \qquad v = -\cos x$

$$= e^{2x}\sin x - \left[-2e^{2x}\cos x + \int 4e^{2x}\cos x \, dx\right]$$

$$\int e^{2x}\cos x \, dx = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x \, dx$$

$$5\int e^{2x}\cos x \, dx = e^{2x}\sin x + 2e^{2x}\cos x$$

divide by 5. Add "+c" at end.

Section 8.2

20. $\int \sin^2 x \cos^3 x \, dx$ odd cosine, factor out $\cos x$, $u = \sin x$

$du = \cos x \, dx$

$$\int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos^2 x}_{1-\sin^2 x \,\to\, 1-u^2} \underbrace{\cos x \, dx}_{du}$$

$$= \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

21. $\int \sin^3 x \cos^3 x \, dx$ ← both odd

① factor out $\sin x$, $u = \cos x$ ←

OR

② factor out $\cos x$, $u = \sin x$

$$\int \underbrace{\sin^2 x}_{1-\cos^2 x \,\to\, 1-u^2} \underbrace{\cos^3 x}_{u^3} \underbrace{\sin x \, dx}_{-du}$$

$u = \cos x$

$du = -\sin x \, dx$

$$\int (1-u^2)u^3(-du)$$

$$- \int (u^3 - u^5)\,du$$

$$= -\frac{u^4}{4} + \frac{u^6}{6} + C$$

$$= \boxed{\frac{-\cos^4 x}{4} + \frac{\cos^6 x}{6} + C}$$

22. $\int \cos^2(4x)\sin^2(4x)\,dx$

all even powers

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$\int \frac{1}{2}(1+\cos(8x))\cdot\frac{1}{2}(1-\cos(8x))\,dx$

$\frac{1}{4}\int\left(1-\cos^2(8x)\right)dx = \frac{1}{4}\int\left(\sin^2(8x)\right)dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos(16x))\,dx$

$= \boxed{\frac{1}{8}\left(x-\frac{1}{16}\sin(16x)\right)+C}$

23. $\int \tan^5 x\sec^3 x\,dx$

① odd tangent, factor out $\sec x\tan x$, $u=\sec x$

$u=\sec x$
$du=\sec x\tan x\,dx$

$\int \underbrace{\tan^4 x}_{}\,\underbrace{\sec^2 x}_{u^2}\,\underbrace{\sec x\tan x\,dx}_{du}$

$\downarrow$
$(\tan^2 x)^2$
$\downarrow$
$(\sec^2 x-1)^2$
$\downarrow$
$(u^2-1)^2$

$\int(u^2-1)^2u^2\,du$

$\int(u^4-2u^2+1)u^2\,du$

$\int(u^6-2u^4+u^2)\,du$

$\frac{1}{7}u^7-2\cdot\frac{u^5}{5}+\frac{u^3}{3}+C \leftarrow u=\sec x$

24. $\int_0^1 \sec^4 x\sqrt{\tan x}\,dx$

even secant, factor out $\sec^2 x$

$u = \tan x$

$$\int_0^1 \underbrace{\sec^2 x}_{\tan^2 x + 1 \,\to\, u^2+1}\underbrace{\sqrt{\tan x}}_{\sqrt{u}}\underbrace{\sec^2 x\,dx}_{du}$$

$u = \tan x$ $\begin{cases} x=1,\ u=\tan(1) \\ x=0,\ u=0 \end{cases}$

$du = \sec^2 x\,dx$

$$= \int_0^{\tan(1)} (u^2+1)\sqrt{u}\,du$$

$$= \int_0^{\tan(1)} \left(u^{\frac{5}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \left(\frac{2}{7}u^{\frac{7}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_0^{\tan(1)}$$

$$= \boxed{\frac{2}{7}(\tan(1))^{\frac{7}{2}} + \frac{2}{3}(\tan(1))^{\frac{3}{2}}}$$

25. $\int \frac{\sin^2(\ln x)}{x}\,dx$

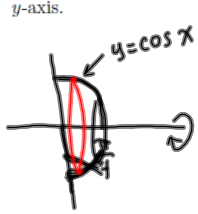
$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \sin^2 u\,du = \int \frac{1}{2}(1-\cos 2u)\,du = \frac{1}{2}\left(u - \frac{1}{2}\sin(2u)\right) + C$$

$$= \frac{1}{2}\left(\ln x - \frac{1}{2}\sin(2\ln x)\right) + C$$

26. Let R be the region bounded by $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$. Find the volume obtained by rotating the region R about the x-axis, then the y-axis.



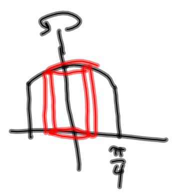
about x-axis

$$V = \int_0^{\frac{\pi}{4}} \pi\cos^2 x\,dx$$

$$= \pi\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(2x))\,dx$$

$$= \frac{\pi}{2}\left(x + \frac{1}{2}\sin(2x)\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{1}{2}(1)\right)$$



about y-axis

shells $r = x$

$h = \cos x$

$$V = \int_0^{\frac{\pi}{4}} 2\pi x\cos x\,dx$$

parts:

$u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \sin x$

$$= 2\pi\left[uv\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} v\,du\right]$$

$$= 2\pi\left[x\sin x\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin x\,dx\right]$$

$$= 2\pi\left[x\sin x + \cos x\right]\Bigg|_0^{\frac{\pi}{4}}$$

$$= \boxed{2\pi\left[\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right]}$$