Spring 2008 Math 152

Overview of Material for Test II

courtesy: Amy Austin

Numerical Integration: sec 8.8

1. Numerical Integration: Suppose I'd like to know $\int_a^b f(x)\,dx$. There are three techniques of approximating an integral:

   I. Midpoint Rule:

   $$\int_a^b f(x)\,dx \approx \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + ... + f(\overline{x_n})]$$

   where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x_i}$ is the midpoint of the *ith* subinterval.

   II. Trapezoid Rule:

   $$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + f(x_n)]$$

   where $\Delta x = \dfrac{b-a}{n}$ and x_i are the points of the partition.

   III. Simpson's Rule:

   $$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + f(x_n)]$$

   where $\Delta x = \dfrac{b-a}{n}$ and x_i are the points of the partition.

   • Error Bound formulas: If you are asked to find an upper bound on the error, these formulas will be provided on the 152 common exam.

   1. Error Bound for Midpoint Rule:

   $$|E_M| \leq \frac{K(b-a)^3}{24n^2}, \text{ where } K = \max|f''(x)| \text{ for } a \leq x \leq b$$

   2. Error Bound for Trapezoid Rule:

   $$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \text{ where } K = \max|f''(x)| \text{ for } a \leq x \leq b$$

   3. Error Bound for Simpson's Rule:

   $$|E_S| \leq \frac{K(b-a)^5}{180n^4}, \text{ where } K = \max|f^{(4)}(x)| \text{ for } a \leq x \leq b$$

Improper Integrals: sec 8.9

2. Improper Integrals:

   Case I: Integrals where one (or both) of the limits is infinite: Your goal is to determine whether the improper integral converges (finite value) or diverges (infinite value).

   a.) $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

   b.) $\displaystyle\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$

   c.) $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$, then try to evaluate both integrals.

   Case II: Integrals where there is a discontinuity on the interval $[a, b]$:

   a.) Suppose $f(x)$ is discontinuous at $x = a$: Then

   $$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

   b.) Suppose $f(x)$ is discontinuous at $x = b$: Then

   $$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

   c.) If $f(x)$ is discontinuous at some c where $a < c < b$, then

   $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, then try to evaluate both integrals.

   • Comparison Theorem for Improper Integrals:

   a.) Suppose $f(x)$ and $g(x)$ are continuous, positive functions on the interval $[a, \infty)$. Also, suppose that $f(x) \geq g(x)$ on the interval $[a, \infty)$. Then:

   (i) If $\int_a^\infty f(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$.

   (Note: If $\int_a^\infty f(x)\,dx$ diverges, no conclusion can be drawn about $\int_a^\infty g(x)\,dx$).

   (ii) If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty f(x)\,dx$.

   (Note: If $\int_a^\infty g(x)\,dx$ converges, no conclusion can be drawn about $\int_a^\infty f(x)\,dx$).

   Note: The way you choose the comparison function: You take the largest part of the numerator over the largest part of the denominator on the interval $[a, \infty)$. Once you find the comparison function, you *must* determine the direction of the inequality, then integrate the comparison function and draw the correct conclusion.

Differential Equations: sec 9.1

3. Def: A differential equation is an equation that contains an unknown function and some of its derivatives. Your primary goal is to try to solve the differential equation.

   • A differential equation is separable if it is in the form $Q(y)dy = P(x)dx$. To solve such an equation, integrate both sides.
   ex: $\frac{dy}{dx} = \frac{4x^2}{2y^4}$: seperate it: $2y^4 dy = 4x^2 dx$, therefore $\frac{2}{5}y^5 = \frac{4}{3}x^3 + C$ Then solve for y. You may have an initial condition which allows you to solve for C: Suppose $y(2) = 3$, then $\frac{2}{5}3^5 = \frac{4}{3}2^3 + C$, solve for C, then solve for y.

   • Brine Problems: Suppose a tank contains L Liters of salt water (could contain no salt at time $t = 0$). Now let's suppose a salt concentration I is going into the tank at a given rate R, the solution is continually stirred and it is exiting the tank at the same rate. Then if $y = y(t)$ is the amount of salt in the tank at time t, then
   $\frac{dy}{dt} = (I) * R - \frac{Y}{L} * R$, and $y(0) =$ amt of salt in the tank at time $t = 0$. Then solve for y.

Differential Equations: sec 9.2

4. Linear differential equations

   • A differential equation is linear if it is in the form
   $\frac{dy}{dx} + P(x)y = Q(x)$. It is important that you recognize which variable is independant and which is dependant. If your equation contains $\frac{dy}{dx}$, then the independant variable is x; the dependant variable is y.

   • To solve a linear differential equation, you must first find the integrating factor $I(x) = e^{\int P(x)\,dx}$.

   • Next, multiply both sides of the differential equation by $I(x)$: $I(x)\left(\frac{dy}{dx} + P(x)y\right) = I(x)Q(x)$, which then becomes
   $(yI(x))' = I(x)Q(x)$. Next, integrate both sides and then solve for y.

Arc Length: sec 9.3

5. There are three possible formulas which gives the length of a curve:

   a.) If $y = f(x)$, $a \leq x \leq b$, then the length of the curve from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

   b.) If $x = g(y)$, $c \leq y \leq d$, then the length of the curve from $y = c$ to $y = d$ is $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

   c.) If $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then the length of the curve from $t = \alpha$ to $t = \beta$ is
   $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area of Revolution: sec 9.4

6. Revolution around the x axis:

   a.) If the curve $y = f(x)$, $a \leq x \leq b$ is revolved around the x axis, then the resulting surface area is given by
   $$SA = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

   b.) If the curve $x = g(y)$, $c \leq y \leq d$ is revolved around the x axis, then the resulting surface area is given by
   $$SA = 2\pi \int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

   c.) If the curve $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, is revolved around the x axis, then the resulting surface area is $SA = 2\pi \int_\alpha^\beta g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

7. Revolution around the y axis:

   a.) If the curve $y = f(x)$, $a \leq x \leq b$ is revolved around the y axis, then the resulting surface area is given by
   $$SA = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

   b.) If the curve $x = g(y)$, $c \leq y \leq d$ is revolved around the y axis, then the resulting surface area is given by
   $$SA = 2\pi \int_c^d g(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

   c.) If the curve $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, is revolved around the y axis, then the resulting surface area is given by
   $$SA = 2\pi \int_\alpha^\beta f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Moments and Centers of Gravity: sec 9.5

8. If we have a system of n particles with masses $m_1, m_2, ..., m_n$ located at the points $x_1, x_2, ..., x_n$ on the x axis, then

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

9. If we have a system of n particles with masses $m_1, m_2, ..., m_n$ located at the points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ in the x-y plane.

   - The **moment of the system about the y axis** is $M_y = \sum_{i=1}^{n} m_i x_i$. This measures the tendency of the system to rotate about the y axis.

   - The **moment of the system about the x axis** is $M_x = \sum_{i=1}^{n} m_i y_i$. This measures the tendency of the system to rotate about the x axis.

   - The **center of mass** is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{M_y}{\sum_{i=1}^{n} m_i} \qquad \textit{and} \qquad \overline{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}$$

10. Now we have a function $y = f(x)$ with uniform density ρ on the interval $[a, b]$.

   - The moment about the y-axis is:

$$M_y = \rho \int_a^b x f(x)\, dx$$

   - The moment about the x-axis is:

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2\, dx$$

   - The x coordinate of the centroid is

$\overline{x} = \dfrac{1}{A} \displaystyle\int_a^b x f(x)\, dx$, where A is the area of the region.

   - The y coordinate of the centroid is

$\overline{y} = \dfrac{1}{A} \displaystyle\int_a^b \frac{1}{2} [f(x)]^2\, dx$, where A is the area of the region.

11. Now we have the region with density ρ bounded by $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ over the interval $[a, b]$.

   - The moment about the y-axis is:

$$M_y = \rho \int_a^b x(f(x) - g(x))\, dx$$

   - The moment about the x-axis is:

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 - (g(x))^2)\, dx$$

   - The x coordinate of the centroid is

$\overline{x} = \dfrac{1}{A} \displaystyle\int_a^b x(f(x) - g(x))\, dx$, where A is the area of the region.

   - The y coordinate of the centroid is

$\overline{y} = \dfrac{1}{A} \displaystyle\int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2)\, dx$, where A is the area of the region.

Hydrostatic Pressure and Force: sec 9.6

12. General formula: The hydrostatic force on a horizontal plate is $F = \rho g d A$ where ρg is the weight density of the liquid, d is the depth of the plate, and A is the area of the plate. If you are finding the force on a vertical plate, finding the hydrostatic force requires integration. Take a small vertical strip of water with area A. Then $F = \displaystyle\int \rho g d A$, where d is the depth of the vertical strip. The limits of integration depends on how you define your axes.