SERIES

courtesy of Amy Austin

Def: Let $\{a_n\} = \{a_1, a_2, a_3, ..., ...\}$ be a sequence. We define the infinite series to be $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... + ...$ In other words, a series is the **sum** of a sequence. The main focus of chapter 10 is to determine when the sum is finite.

Def: Let $\sum_{n=1}^{\infty} a_n$ be a series. We will construct the sequence of partial sums

$\{s_n\} = \{s_1, s_2, s_3, ..., ...\}$ as follows:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

Therefore a general formula for s_n is

$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n.$

If $\lim_{n\to\infty} s_n = s$, where s is finite, then we say the series $\sum_{n=1}^{\infty} a_n$ converges and it's **sum** is s. If $\lim_{n\to\infty} s_n$ is infinite or does not exist, then we say the series $\sum_{n=1}^{\infty} a_n$ diverges.

Test for Convergence

Below are the various tests to determine whether a particular series converges or diverges.

1. **The Test for Divergence**: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. NOTE: The converse is not necessarily true: If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ does not necessarily converge. Therefore if you find that $\lim_{n\to\infty} a_n = 0$, then the divergence test fails. For example the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, however the TERMS $\{\frac{1}{n}\}$ do go to zero-just not fast enough to get a finite SUM.

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2. **Geometric series**: The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If $|r| < 1$, then the sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

3. **The Integral Test**: If $f(x)$ is a positive, continuous, decreasing function on $[1, \infty]$, and $a_n = f(n)$. Then:

   a.) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ converges.

   b.) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

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4. The **p-series** $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

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5. **The Comparison Test**: (Use this test if the series is a series of **positive** terms, and the series is comparable to a p-series or a geometric series.)
   Suppose $\sum_{n=1}^\infty a_n$ and $\sum_{i=1}^\infty b_n$ are series of positive terms.

   a.) If $\sum_{n=1}^\infty b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^\infty a_n$ is also convergent.

   b.) If $\sum_{n=1}^\infty b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^\infty a_n$ is also divergent.

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6. **The Limit Comparison Test**: Conditions for using this test are the same conditions as the comparison test.
   Suppose $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ are series of positive terms.

   a.) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.

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7. **The Alternating Series Test**: If the alternating series $\sum_{n=1}^\infty (-1)^n a_n$ satisfies

   a.) $a_{n+1} \leq a_n$ for all n (ie the sequence $\{a_n\}$ is decreasing).

   b.) $\lim_{n\to\infty} a_n = 0$

   then the series converges.

8. **The Ratio Test**: (Use this test if the series contains $n!$ or numbers raised to the nth power, such as 2^n. If the **ONLY** number raised to the nth power is $(-1)^n$, then use the alternating series test).

   a.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

   b.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   c.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test fails.

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9. **Remainder formulas:**

- **The Remainder Estimate for the Integral test**: Suppose $\sum_{n=1}^{\infty} a_n$ is a series which was shown to be convergent as a result of the integral test or a comparison test. This means that the sum of the series is finite. Let's say $\sum_{n=1}^{\infty} a_n = s$. Suppose further that I used a partial sum $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$ to approximate s. Then the remainder is defined to be $R_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + ... + ...$

   a.) If we want to get an upper bound for the error in using s_n to approximate s, then $R_n \leq \int_n^{\infty} f(x)\,dx$.

   b.) If we want to get an interval on which the remainder lies, then

   $$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx.$$

- **The Alternating Series Theorem**: If $\sum_{n=1}^{\infty} (-1)^n a_n$ is a convergent alternating series, and I used a partial sum $s_n = \sum_{i=1}^{n} (-1)^i a_i$ to approximate the sum, then an upper bound on the absolute value of the remainder is $|R_n| \leq a_{n+1}$.