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1. Since $c_n = \frac{f^{(n)}(-4)}{n!}$, $c_n = \frac{(-2)^n}{7^n(n+5)}$

c_n is the coefficient of $(x+4)^n$.

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{7^n(n+5)}(x+4)^n$$

2.

n	$f^{(n)}(x)$	$f^{(n)}(3)$	$c_n = \frac{f^{(n)}(3)}{n!}$
0	$\frac{1}{x}$	$\frac{1}{3}$	$c_0 = \frac{1}{3}$
1	$\frac{-1}{x^2}$	$-1/9$	$c_1 = -1/9 = \frac{-1}{3^2}$
2	$\frac{(-1)(-2)}{x^3}$	$\frac{2}{3^3}$	$c_2 = \frac{2}{3^3 \cdot 2} = \frac{1}{3^3}$
3	$\frac{(-1)(-2)(-3)}{x^4}$	$\frac{-3!}{3^4}$	$c_3 = \frac{-3!}{3^4 \cdot 3!} = \frac{-1}{3^4}$

etc. $c_n = \frac{(-1)^n}{3^{n+1}}$

Series: $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-3)^n$

The 10.6 way: $\frac{1}{x} = \frac{1}{3+(x-3)} = \frac{1}{3(1+\frac{x-3}{3})}$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-3}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{3^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{3^{n+1}}$$

3.

n	$f^{(n)}(x)$	$f^{(n)}(3)$	$c_n = \frac{f^{(n)}(3)}{n!}$
0	xe^x	$3e^3$	$c_0 = 3e^3$
1	$e^x + xe^x$	$4e^3$	$c_1 = 4e^3$
2	$e^x + e^x + xe^x = 2e^x + xe^x$	$5e^3$	$c_2 = \frac{5e^3}{2!} = \frac{5e^3}{2}$
3	$2e^x + e^x + xe^x$	$6e^3$	$c_3 = \frac{6e^3}{3!}$

$$f^{(n)}(x) = ne^x + xe^x$$

$$f^{(n)}(3) = ne^3 + 3e^3 = (n+3)e^3$$

$$\sum_{n=0}^{\infty} (n+3)e^3 \frac{(x-3)^n}{n!}$$ Find R.O.C.

Ratio Test $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+4)\cancel{e^3}}{(n+1)!}|x-3|^{n+1} \cdot \frac{n!}{(n+3)\cancel{e^3}|x-3|^n}$

$$= \frac{(n+4)}{(n+1)(n+3)}|x-3| \xrightarrow[n\to\infty]{} 0$$

$\left(\text{Since } \frac{n+4}{(n+1)(n+3)} \xrightarrow[n\to\infty]{} 0\right)$ for any x.

$R = \infty$

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	e^x	1	$c_0 = 1$
general			
n	e^x	1	$c_n = \frac{1}{n!}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$ Find ROC:

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \xrightarrow[n \to \infty]{} 0 \text{ for any } x.$$

$R = \infty$

5. $\sin x = f(x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\sin x$	0	$c_0 = 0$
1	$\cos x$	1	$c_1 = 1$
2	$-\sin x$	0	$c_2 = 0$
3	$-\cos x$	-1	$c_3 = -\frac{1}{3!}$

repeat this pattern.
All even powers have 0 coefficients, odd powers alternate in sign.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ $R = \infty$

$f(x) = \cos x$

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\cos x$	1	$c_0 = 1$
1	$-\sin x$	0	$c_1 = 0$
2	$-\cos x$	-1	$c_2 = \frac{-1}{2!}$
3	$\sin x$	0	$c_3 = 0$

repeat this

All odd powers have 0 coefficients.
The even powers alternate in sign.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R = \infty$$

7. a) $$f(x) = \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

b) $$xe^{-x} = x\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \cancel{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$$

8. $$\int \frac{\sin(2x)}{x}\,dx = \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}\,dx$$

$$= \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}\,\frac{x^{2n+1}}{(2n+1)} + C$$ by the Thm.

9. $$\int_0^{.5} \cos(x^2)\,dx = \int_0^{.5}\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n}}{(2n)!}\,dx$$

$$= \int_0^{.5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,\frac{x^{4n+1}}{(4n+1)}\Bigg|_0^{.5}$$

Using alternating series remainder, which is valid in $[0, .5]$ (x^{4n+1} cannot change sign anyway)

$$= \sum_{n=0}^{\infty}\frac{(-1)^n (\frac{1}{2})^{4n+1}}{(2n)!(4n+1)}$$

Find n so $R_n \le 10^{-3}$ using the alternating series remainder $|R_n| \le b_{n+1} = \frac{1}{2^{4n+5}(2n+2)!(4n+5)}$

Solve $\dfrac{1}{2^{4n+5}(2n+2)!(4n+5)} \le 10^{-3}$

Try $n=1$ $\dfrac{1}{2^{9}(4!)9}$ works

$S_1 = \dfrac{1}{2} + \dfrac{(-1)\ 1/2^5}{2!(5)} = \dfrac{1}{2} - \dfrac{1}{320}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{3n}}{n!}$ What is raised to the power n?

$(-1)^n 2 \cdot 2^n (x^3)^n = 2(-2x^3)^n$

The series is

$\displaystyle\sum_{n=0}^{\infty} 2 \frac{(-2x^3)^n}{n!} = 2e^{-2x^3}$

11. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n}(2n)!}$ looks like cos since $(2n)!$ is in the denominator.

$\displaystyle\sum_{n=0}^{\infty} (-1)^n \pi \cdot \frac{\pi^{2n}}{3^{2n}} \frac{1}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi \cdot \left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$

$= \pi \cos(\pi/3) = \frac{1}{2}\pi$

12. $5 + \frac{25}{2} + \frac{125}{3!} + \frac{625}{4!} + \cdots$

The numerators are $5^1, 5^2, 5^3, 5^4, \ldots$

$$\sum_{n=1}^{\infty} \frac{5^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n}{n!} - 1 = e^5 - 1$$

(The $n=0$ term is missing from e^5)

13. $$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Find the coefficient of x^{20}.
$2n = 20$ if $n = 10$ so the coefficient is $\frac{1}{10!} = c_{20} = \frac{f^{(20)}(0)}{20!}$

$$\boxed{\frac{20!}{10!} = f^{(20)}(0)}$$

14.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$	c_n
0	$\cos x$	$\cos\frac{\pi}{3} = \frac{1}{2}$	$c_0 = \frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$	$c_1 = -\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$	$c_2 = -\frac{1}{4}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$	$c_3 = \frac{\sqrt{3}}{12}$

$$T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3$$

15. Find $T_2(x)$ for $f(x) = \ln x$ about $a = 2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	c_n
0	$\ln x$	$\ln 2$	$c_0 = \ln 2$
1	$\frac{1}{x}$	$\frac{1}{2}$	$c_1 = \frac{1}{2}$
2	$\frac{-1}{x^2}$	$-1/4$	$c_2 = -1/8$

$$T_2(x) = \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2)^2$$

Estimate $|\ln x - T_2(x)|$ on $[1, 3.2]$

$$\left|f^{(3)}(x)\right| = \left|\frac{2}{x^3}\right| \leq 2 \quad \text{on } [1, 3.2]$$

$$|\ln x - T_2(x)| \leq \frac{2\,|x-2|^3}{3!} = \frac{1}{3}|x-2|^3$$

$$\leq \frac{(1.2)^3}{3}$$