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On local stability for a nonlinear difference equation with a non-hyperbolic equilibrium and fading stochastic perturbations

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We consider the nonlinear stochastic difference equation

$$X_{n+1} = X_n - f(X_n) + \sigma_n \xi_{n+1}, \quad n = 0, 1, \ldots, \ X_0 \in \mathbb{R}.$$

Here, $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables with zero mean and unit variance and with distribution functions F_n. The function $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$, $xf(x) > 0$ for $x \neq 0$. We establish a condition on the noise intensity σ and the rate of decay of the tails of the distribution functions F_n, under which the convergence of solutions to zero occurs with probability zero. If this condition does not hold, and f is bounded by a linear function with slope $2 - \gamma$, for $\gamma \in (0, 2)$, all solutions tend to zero a.s. On the other hand, if f grows more quickly than linear function with slope $2 + \gamma$, for $\gamma > 0$, the solutions tend to infinity in modulus with arbitrarily high probability, once the initial condition is chosen sufficiently large. Such equations can still demonstrate local stability; for a wide class of highly nonlinear f, it is shown that solutions tend to zero with arbitrarily high probability, once the initial condition is chosen appropriately. Results which elucidate the relationship between the rate of decay of the noise intensities and the rate of decay of the tails, and the necessary condition for stability, are presented. The connection with the asymptotic dynamical consistency of the system, when viewed as a discretisation of an Itô stochastic differential equation, is also explored.

Keywords: stochastic difference equation; asymptotic stability; local and global stability; almost sure asymptotic stability; non-hyperbolic equilibrium

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1. Introduction

The main contribution in this paper is to allow us to determine stability and instability of solutions of stochastic difference equations

$$X_{n+1} = X_n - f(X_n) + \sigma_n \xi_{n+1}, \quad n = 0, 1, \ldots, \ X_0 \in \mathbb{R},$$

making minimal assumptions on f, σ_n and the distribution functions F_n of ξ_n. We make the assumption that $xf(x) > 0$ for all $x \neq 0$, that $f(0) = 0$ and that $\inf_{|x|\geq c}|f(x)| > 0$. These assumptions ensure that the solution tends to revert towards the unique equilibrium value

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of the unperturbed deterministic difference equation. In studying stability, we also assume that $\sigma_n \to 0$ as $n \to \infty$.

These assumptions are suitable for modelling the essential mechanism underlying dynamic equilibrium in a highly schematised model of a self-regulating economic system which is subjected to persistent stochastic shocks, whose intensity ultimately fades over time. The behaviour of the system in the pre-shock epoch is given by the deterministic equation

$$X_{n+1} = X_n - f(X_n), \quad n < 0.$$

Our results show that if the system is globally stable before the shock, the system will return to equilibrium, provided the shock fades sufficiently quickly. However, it transpires that, no matter how effective the self-regulatory property of the system, if the shock fades out more slowly than a critical rate, then the system will not be stable. The critical rate depends on the 'fatness' of the tails of the shock distributions. Therefore, as heavy-tailed shocks abound in financial systems, even a fading shock intensity presents a danger to the system.

For equations which are only locally stable in the absence of such shocks, the potential exists for the shock to push the system into an unstable region. It is notable that the type of instability exhibited by the equations studied in this paper is associated with overshooting across (oscillation about) the equilibrium. Such overshooting is typical of economic systems under external stress. Therefore, the conditions we impose provide a test of the robustness of the equilibrium mechanism under persistent perturbations.

It is important to note that our results are derived with a non-linearisable equilibrium in mind. Thus, while the analysis holds for linear equations as well, we do not use any linearisation techniques close to equilibrium. On the other hand, we prove results when f does not obey a global linear bound. Therefore, we are not bound to linear hypotheses or methods for large values of the state space either.

In earlier work [1], stability results were proven under the assumption that σ is a square summable sequence. However, as we show here, this condition is unnecessary. Instead, it is only required that $\lim_{n\to\infty}\sigma_n\xi_{n+1} = 0$ a.s., which can be reduced to a weaker conditions on σ and the distribution functions. The method of proof departs completely from the semimartingale convergence technique. Instead, we rely upon viewing the stochastic term as a perturbation of a deterministic equation, where the perturbation has known asymptotic behaviour. The conditions for global stability that we obtain are likely to be relatively sharp, as it is shown that they are almost necessary and sufficient in the deterministic case. Our method of proof also enables us to determine necessary and sufficient conditions on the noise perturbation for asymptotic stability. Moreover, by variation on standard Borel–Cantelli arguments, we are able to obtain conditions relating the rate of decay of the noise intensity to the rate of decay of the tails of the distribution functions. We show how fast σ needs to decay to compensate slowly decaying tails, and vice versa.

The paper is organized as follows. In Section 2, we give necessary definitions from stochastic analysis, formulate main assumptions and discuss results of the paper. Section 3, is devoted to the global stability as well as instability for deterministic difference nonhomogeneous equations. In Section 4, we prove local stability of solution of deterministic difference nonhomogeneous equations with general nonlinear function. In Section 5, we prove global and local stability results for stochastic difference equations, and in Section 6, we deduce stability conditions for stochastic equations with a small parameter, and show how these results can be used to show that, with a sufficiently small mesh size, the stochastic difference equation successfully mimics the asymptotic behaviour of the Itô equation of which it is a discretisation. In Section 7, we derive necessary and sufficient conditions on σ_n and the tails of the distributions F_n which guarantee that $\sigma_n\xi_{n+1} \to 0$. In Section 8, we give several technical proofs postponed from the main part of the paper.

2. Definitions, assumptions and discussion of the results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. We suppose that

ASSUMPTION 2.1. $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence of independent, continuously distributed random variables with distribution functions* F_n, supp $F_n = (-\infty, \infty)$, *and with* $\mathbb{E}\xi_n = 0$, $\mathbb{E}\xi_n^2 = 1$.

We suppose that filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated, namely that $\mathcal{F}_n = \sigma\{\xi_0, \xi_1, \dots \xi_n\}$. Among all sequences $(X_n)_{n\in\mathbb{N}}$ of random variables we distinguish those for which X_n are $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. We use the standard abbreviation 'a.s.' for the wordings 'almost sure' or 'almost surely' with respect to the fixed probability measure $\mathbb{P}$ throughout the text. For more details on stochastic concepts and notations, consult Ref. [11].

Let $\zeta \in \mathbb{R}$ be arbitrary. We consider the nonlinear stochastic difference equation

$$X_{n+1} = X_n - f(X_n) + \sigma_n\xi_{n+1}, \quad n = 0, 1, \dots; \; X_0 = \zeta. \tag{1}$$

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, obeying the following properties:

$$uf(u) > 0, \quad u \neq 0, \text{ for all } u \in \mathbb{R}, \; f(0) = 0; \tag{2}$$

$$\inf_{u>c}|f(u)| > 0 \quad \text{for all } c > 0; \tag{3}$$

$$|f(u)| \leq (2-\gamma)|u|, \quad \gamma \in (0, 2). \tag{4}$$

In this paper, we focus on the following questions.

(i) What are the conditions on f and σ_n which ensure that for all initial values $X_0 \in \mathbb{R}$

$$\lim_{n\to\infty} X_n = 0, \quad \text{a.s.?} \tag{5}$$

(ii) Can we relax the condition (4), but still have (5) fulfilled for some initial values?

(iii) What are the least restrictive conditions on σ_n which ensure

$$\lim_{n\to\infty} \sigma_n\xi_{n+1} = 0, \quad \text{a.s.?} \tag{6}$$

First, we show in Theorem 5.2 that (1) is almost surely *not* asymptotically stable if (6) does not hold. Therefore, if we are interested in asymptotic stability, we must have (6). The independence of the random variables ξ ensures that (6) is equivalent to

$$\sum_{n=1}^{\infty}\left[1 - F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right) + F_{n+1}\left(-\frac{\varepsilon}{|\sigma_n|}\right)\right] < \infty \quad \text{for all } \varepsilon \in \mathbb{R}^+, \tag{7}$$

as proven in Lemma 5.1.

To answer question (i) we show in Theorem 5.3 that when f obeys conditions (2)–(4) and $(X_n)_{n\in\mathbb{N}}$ is a solution to equation (1) with arbitrary initial condition $X_0 = \zeta$, then (5) is equivalent to (6) which is in turn equivalent to (7). The proof of this fact is based on the similar result for deterministic difference equation

$$x_{n+1} = x_n - f(x_n) + S_n, \quad n = 1, 2, \ldots. \tag{8}$$

Namely, if f obeys (2)–(4), $(x_n)_{n\in\mathbb{N}}$ is a solution of (14) with arbitrary initial condition $x_0 \in \mathbb{R}$ and

$$\lim_{n\to\infty} S_n = 0, \tag{9}$$

it is shown in Lemma 3.1 that $x_n \to 0$ as $n \to \infty$. In fact, it is shown in Lemma 3.2 that the assumptions (2)–(4) as well as (9) are close to necessary for the global stability of solutions of (8).

With regard to question (ii), we first obtain results on local stability for solutions of deterministic difference equation with small parameter

$$x_{n+1} = x_n - hf(x_n) + \sqrt{h} S_n, \quad n = 0, 1, \ldots, \tag{10}$$

and a general nonlinear function f, which can grow polynomially, exponentially, or super-exponentially in such a way that $\limsup_{|x|\to\infty} f(x)/x = \infty$. Then, under the same conditions on f, we prove in Theorem 6.1 a local stability result for the stochastic difference equation with small parameter

$$X_{n+1} = X_n - hf(X_n) + \sqrt{h}\sigma_n \xi_{n+1}, \quad n = 0, 1, \ldots, X_0 = \zeta. \tag{11}$$

In Theorem 5.5, it is shown that whenever $\liminf_{|x|\to\infty} f(x)/x > 2$, solutions of (1) are unstable with arbitrarily high probability, once $|\zeta|$ is sufficiently large. Taken together Theorems 5.5 and 6.1 show that solutions of a large class of highly nonlinear stochastic difference equations of the form (11) are unstable with arbitrarily large probability if the initial condition is sufficiently large, and asymptotically stable with arbitrarily large probability if the initial conditions are chosen from some zero-neighborhood.

To the best of our knowledge this is a first local stability result for stochastic difference equation in the literature, at least in the case when f is such a general nonlinear function as stipulated in Lemma 4.6 below. However, the question of (global) stability of stochastic difference equations has been actively studied in recent years. Some representative papers are [1–3,5–10].

To answer question (iii), we establish the connection between the rate of decay of σ_n and asymptotic behaviour of the tails of probability distribution functions F_n of the random variables ξ_n.

In Proposition 7.2, we give a prescription that for a sequence $(\xi_n)_{n\in\mathbb{N}}$ of random variables satisfying Assumption 2.1 constructs a deterministic positive sequence $(r_n)_{n\in\mathbb{N}}$ such that (6) is true whenever

$$\limsup_{n\to\infty} \frac{\log|\sigma_n|}{\log r_n} < -1. \tag{12}$$

The condition (12) is close to being necessary to ensure (6): we prove that (6) implies that

$$\liminf_{n\to\infty} \frac{\log|\sigma_n|}{\log r_n} \le -1.$$

When $1 - F_n$ decays exponentially and super-exponentially with the exponent $-p(x)$, we show in Proposition 7.6 that (6) holds provided $\sigma_n p^{-1}(\log n) \to 0$ as $n \to \infty$. Moreover, if $n \mapsto |\sigma_n|$ is non-decreasing, we have that (6) implies $\sigma_n p^{-1}(\log n) \to 0$ as $n \to \infty$. This result enables us to prove that when ξ is a sequence of independent and identically distributed (iid) standard normal random variables then $\sigma_n^2 \log n \to 0$ is a necessary and sufficient condition to guarantee the global stability of solutions of (1) under the conditions (2)–(4) for example. We also employ this result and Theorem 5.3 in Theorem 6.2 to show that an Euler–Maruyama discretisation of the asymptotically stable stochastic differential equation

$$\mathrm{d}X(t) = -f(X(t))\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) \tag{13}$$

tends to zero almost surely if the mesh size $h < h_0$, where h_0 can be determined *a priori*. Interestingly, the condition $\sigma^2(t) \log t \to 0$ is also necessary and sufficient for the stability of equation (13), as was shown in Chan and Williams [4].

We also prove that when $1 - F_n(y)$ decays more quickly then y^{-2}, there exists σ_n such that $\sum_{i=1}^{\infty} \sigma_i^2 = \infty$, but $\lim_{n\to\infty} X_n = 0$, a.s., where X_n is a solution of equation (1) with this particular σ_n. Thus, we improve a result of Ref. [1], where $\lim_{n\to\infty} X_n = 0$, a.s., was proved under assumption that $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$.

In the course of the paper we discuss four types of tails of distribution of ξ_n: (i) polynomial; (ii) sub-exponential but super-polynomial; (iii) exponential and (iv) super-exponential. To illustrate our results, examples from these types of distributions are given at the end of subsections 7.1, 7.2 and 7.4.

3. Global stability and instability for perturbed deterministic difference equations

In this section, we present necessary and sufficient conditions for the asymptotic stability and instability of the zero equilibrium of

$$x_{n+1} = x_n - f(x_n) + S_n, \quad n = 1, 2, \ldots, x_0 \in \mathbb{R}. \tag{14}$$

The hallmark of results in this section is that it is either explicitly or tacitly assumed that the nonlinear function f obeys a global linear bound. This assumption is removed in the next section.

We first prove a result on the global stability of solutions of (14) when all we require of the perturbation S_n is that it tends to zero as $n \to \infty$.

LEMMA 3.1. *Suppose that f obeys* (2)–(4) *and $S_n \to 0$, $n \to \infty$. Let $(x_n)_{n\in\mathbb{N}}$ be a solution of* (14) *with arbitrary initial condition $x_0 \in \mathbb{R}$. Then $x_n \to 0$.*

Proof. We let $g(x) = x - f(x)$ and prove the following two properties of g

$$|g(x)| \le |x| \quad \text{for all } x \in \mathbb{R}, \tag{15}$$

$$\text{for all } M > 0 \text{ there exists } \delta(M) \text{ such that } |x| \ge M \Rightarrow |g(x)| \le |x| - \delta(M). \tag{16}$$

To prove (15), we note that by (4) for $x > 0$

$$-(1-\gamma)|x| = -(1-\gamma)x \le g(x) = x - f(x) < x = |x|,$$

while for $x < 0$

$$-|x| = x < g(x) = x - f(x) \le -(1-\gamma)x = (1-\gamma)|x|.$$

To prove (16), we fix some $M > 0$ and define $\kappa(M) = \inf_{|u|>M}|f(u)|$. Then for $x \ge M$

$$-|x| + \gamma M \le -|x| + \gamma|x| \le g(x) = x - f(x) \le x - \kappa(M) = |x| - \kappa(M),$$

while for $x \le -M$

$$|x| - \gamma M \ge |x| - \gamma|x| \ge g(x) = x - f(x) \ge -|x| + \kappa(M).$$

Thus the property (16) holds true for

$$\delta(M) = \min\{\kappa(M), \gamma M\}. \tag{17}$$

We fix some $M > 0$ and define $\delta(M)$, satisfying (17). We find $N_M \in \mathbb{N}$ such that $\sup_{n \ge N_M}|S_n| < \delta(M)$. We put

$$\varepsilon = \sup_{n \ge N_M}|S_n|.$$

Properties (15) and (16) imply the following:

- if $|x_n| < M$ for some $n \ge N_M$, then $|x_{n+1}| < M + \varepsilon$;
- if $|x_n| \ge M$ for some $n \ge N_M$, then $|x_{n+1}| < |x_n| - (\delta(M) - \varepsilon)$.

From the above we obtain that

- if $|x_n| < M + \varepsilon$ for $n \ge N_M$, then $|x_{n+1}| < M + \varepsilon$;
- if $|x_n| > M$ for some $n \ge N_M$, then there exists $k > n$ such that $|x_k| < M + \varepsilon$.

Thus for any initial value x_0, sequence $(x_n)_{n\in\mathbb{N}}$ eventually gets into the interval $[-M - \varepsilon, M + \varepsilon]$ and stays there. Since M and ε can be chosen arbitrary small, this means that $\lim_{n\to\infty}x_n = 0$. □

It transpires that the assumptions of Lemma 3.1 are close to necessary for the global stability of solution to equation (14).

LEMMA 3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to* (14) *where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous function, $f(0) = 0$.*

(i) *If $x_n \to 0$ for some initial condition $x_0 \in \mathbb{R}$, then $S_n \to 0$.*
(ii) *If $S_n \equiv 0$ and $x_n \to 0$ for any initial condition $x_0 \in \mathbb{R}$ then $uf(u) > 0$ for all $u \ne 0$.*
(iii) *If $x_n \to 0$ for any initial condition $x_0 \in \mathbb{R}$ and $f(-u) = -f(u)$, then $|f(u)| < 2|u|$ for all $u \ne 0$.*
(iv) *For any $S_n \downarrow 0$ with $\sum_{i=1}^{\infty}S_i = \infty$, we can construct a function $f : \mathbb{R} \to \mathbb{R}$, $xf(x) > 0$ for $x \ne 0$, $f(0) = 0$ and $f(u) \to 0$ when $u \to 0$, such that corresponding equation* (14) *has a solution $(x_n)_{n\in\mathbb{N}}$, which tends to infinity for some initial value $x_0 > 0$.*

Example 3.3. Part (iii) of Lemma 3.2 does not hold without the assumption that $f(-u) = f(u)$. Indeed, let $S_n \geq 0$, $S_n \to 0$, and

$$f(x) = \begin{cases} x/2 & x > 0; \\ 2x & x \leq 0. \end{cases}$$

Then $\lim_{n\to\infty} x_n = 0$, where $(x_n)_{n\in \mathbf{N}}$ is a solution to (14) with arbitrary initial value $x_0 \in \mathbb{R}$ and function f, defined above. To prove this we note that if $x_0 > 0$, then $x_{n+1} = (1/2)x_n + S_n > 0$ for all $n = 0, 1, 2, \ldots$, while when $x_0 < 0$, then $x_1 = -x_0 + S_1 > 0$.

We now show that when the condition (4) on f is dropped and instead we assume that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 2, \tag{18}$$

then the solution of equation (14) with sufficiently large initial value does not tend to zero.

Lemma 3.4. *Let condition* (18) *hold. Suppose also that for some* $\bar{S} > 0$ *and for all* $n \in \mathbb{N}$

$$|S_n| \leq \bar{S}. \tag{19}$$

Let $(x_n)_{n\in\mathbb{N}}$ *be a solution of* (14) *with initial condition* $x_0 \in \mathbb{R}$. *Then there is a* $d > 0$ *such that* $\lim_{n\to\infty}|x_n| = \infty$ *when* $|x_0| > d$. *Moreover,* $\limsup_{n\to\infty} x_n = \infty$ *and* $\liminf_{n\to\infty} x_n = -\infty$.

Proof. Condition (18) implies that there exist $\gamma > 0$ and $d_1 > 0$ such that for $|u| > d_1$

$$|f(u)| \geq \left(2 + \frac{\gamma}{2}\right)|u|.$$

We define $d = \max\{d_1, 4\bar{S}/\gamma\}$. For all $x_n > d$ we have

$$\begin{aligned} x_{n+1} &< x_n - \left(2 + \frac{\gamma}{2}\right)x_n + S_n = -\left(1 + \frac{\gamma}{4}\right)x_n - \frac{\gamma}{4}x_n + S_n \\ &\leq -\left(1 + \frac{\gamma}{4}\right)x_n - \frac{\gamma d}{4} + \bar{S} \leq -\left(1 + \frac{\gamma}{4}\right)x_n, \end{aligned}$$

while for $x_n < -d$ we can similarly show that $x_{n+1} > -(1 + (\gamma/4))x_n$. Thus in both cases $|x_{n+1}| \geq (1 + (\gamma/4))|x_n|$, which easily implies that $\lim_{n\to\infty}|x_n| = \infty$. From the above estimates we also conclude that x_n changes sign at each step. □

4. Local stability for deterministic difference equations

In this section, we obtain a local stability result for difference equation with small parameter h

$$x_{n+1} = x_n - hf(x_n) + \sqrt{h}S_n, \quad n = 0, 1, 2, \ldots, x_0 \in \mathbb{R}. \tag{20}$$

Herein, it is implicitly assumed that due to the presence of a small parameter h multiplying f, it is only worthwhile to consider the case in which no global linear growth condition is imposed on f, i.e. the case in which

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} = \infty. \tag{21}$$

This is the crucial distinction between the assumptions used in stability proofs in this in section, and those employed in the last section.

In the following subsection 4.1, sufficient conditions which guarantee local stability are stated and discussed, and the main result on local stability stated explicitly. We show also that the presence of a small parameter in highly nonlinear nonhomogeneous equation (20) is essential in order to guarantee that $x_n \to 0$ as $n \to \infty$ for all initial values x_0 from some neighbourhood of zero. The proofs of a supporting lemma and the main result are given in subsection 4.3.

4.1 Discussion of hypotheses and statement of main result

Despite the fact that (21) is implicitly in force throughout this section, we still demand that f obeys a *local* linear growth condition: that is, we assume that there exist $\varsigma > 0$ and $K > 0$ such that

$$|f(x)| \le K|x|, \quad \text{for all } x \in (-\varsigma, \varsigma). \tag{22}$$

We also assume that for every $a > 0$

$$\begin{aligned} &\lim_{u\to\infty} \sqrt{\frac{u}{f(u)}} \inf_{s\in\left(0,a\sqrt{u/f(u)}\right)} |f(u-s)| = \infty, \\ &\lim_{u\to-\infty} \sqrt{\frac{u}{f(u)}} \inf_{s\in\left(0,a\sqrt{u/f(u)}\right)} |f(u+s)| = \infty. \end{aligned} \tag{23}$$

Condition (23) can be considered a generalization of condition (3). As shown in examples in this section, it is fulfilled for many important types of function f, including polynomial, exponential, and super-exponential growing functions. Moreover, (23) holds for functions f with positive derivatives, which increase when $|x| \to \infty$, and which have the property that $f'(x)/|f(x)|^{1+\varepsilon}$ has a limit as $|x| \to \infty$; indeed this limit value can only be zero.

The main result of this section is the following.

THEOREM 4.1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *obeys* (2), (22), (23) *and* $S_n \to 0$, $n \to \infty$. *Then there is an* $h_0 > 0$ *such that for every* $h \le h_0$ *there exist* $L(h)$, $R(h) > 0$ *such when* $x_0 \in [-L(h), R(h)]$, *the solution* $(x_n)_{n\in\mathbb{N}}$ *of* (14) *obeys the property:* $\lim_{n\to\infty} x_n = 0$.

In advance of proving Theorem 4.1, we discuss the conditions (22) and (23).

Firstly, we show that local stability will not necessary hold without the condition (22). To see this, consider the equation

$$x_{n+1} = x_n - h\sqrt{|x_n|}\,\mathrm{sgn}\,(x_n). \tag{24}$$

In this case, $f(x) = \mathrm{sgn}\,(x)\sqrt{|x|}$ does not obey (22). Now, if $x_0 \in (-h^2/4, h^2/4)$, it can be verified that $|x_{n+1}| > |x_n|$, and that $|x_n| < h^2/4$ for all $n \in \mathbb{N}$. From this it can be concluded that $|x_n| \to h^2/4$ as $n \to \infty$, with the solution converging to the 2-cycle $\{-h^2/4, h^2/4\}$.

Therefore, for any arbitrarily small initial condition, the solution does not converge to zero as $n \to \infty$.

Next, by imposing some monotonicity on f, we can deduce sufficient conditions on f which are moreover readily verified than (23), but which nonetheless imply (23).

LEMMA 4.2. *If f obeys*

$$|f'(x)| \quad \textit{increases}\text{ as } \ |x| \to \infty, \tag{25}$$

and for some $\varepsilon < 1/2$

$$\lim_{|x|\to\infty} \frac{f'(x)}{|f(x)|^{1+\varepsilon}} = \kappa \in [0, \infty], \tag{26}$$

then (23) *holds true.*

If f obeys (26), then the limit κ in (26) can be only zero, and therefore, for some H, $K > 0$

$$\frac{f'(x)}{|f(x)|^{1+\varepsilon}} < K, \quad \text{for all } \ |x| > H. \tag{27}$$

This can be established by writing (26) as a differential inequality, and by then showing that this differential inequality must have an exploding solution, so that f is not defined on all of $\mathbb{R}$, contradicting the existence of f on $\mathbb{R}$.

The example below shows that the condition (23) holds true for many important highly non-linear functions.

Example 4.3. Let a, b, c, $m > 0$. Then Lemma 4.2 implies that any of the following functions $f_j : \mathbb{R} \to \mathbb{R}$ satisfies (23), where

$$f_1(x) = a \operatorname{sgn}(x)|x|^m, \quad f_2(x) = \operatorname{sgn}(x)\, \mathrm{e}^{a|x|^m}, \ f_3(x) = \operatorname{sgn}(x)\, \mathrm{e}^{c\mathrm{e}^{b\mathrm{e}^{a|x|}}}, \ \ldots,$$

and by the same token $f(x) = \operatorname{sgn}(x) \exp\{\exp\{\ldots\{\exp a|x|\}\}\}$ where there can be any finite number of compositions of the exponents.

In the following examples, where no monotonicity is imposed, we can establish condition (23) by direct estimation. The calculations are tedious and hence omitted.

Example 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $K > k$ and one of the following conditions hold true.

(i) There exist $1 < m < n < 2m + 1$, such that

$$ku^m \le f(u) \le Ku^n, \quad \text{for all } \ u \ge 0,$$

$$-Ku^n \le f(u) \le -ku^m, \quad \text{for all } \ u < 0.$$

(ii) There exist $n > 0$, $0 < b \leq c < 2b$, such that

$$ku\mathrm{e}^{bu^n} \leq f(u) \leq Ku\mathrm{e}^{cu^n}, \quad \text{for all } u \geq 0,$$

$$-ku\mathrm{e}^{bu^n} \geq f(u) \geq -Ku\mathrm{e}^{cu^n}, \quad \text{for all } u < 0.$$

(iii) There exist $0 < b \leq c < 2b$, such that

$$\begin{aligned} &ku\mathrm{e}^{b\mathrm{e}^u} \leq f(u) \leq Ku\mathrm{e}^{c\mathrm{e}^u}, \quad \text{for all } u \geq 0, \\ &-ku\mathrm{e}^{b\mathrm{e}^u} \geq f(u) \geq -Ku\mathrm{e}^{c\mathrm{e}^u}, \quad \text{for all } u < 0. \end{aligned} \tag{28}$$

Then condition (23) holds.

To prove local stability we establish in Lemma 4.6 the existence of an interval containing zero and the initial condition such that the solution $(x_n)_{n\in\mathbb{N}}$ remains in the interval. Afterwards, in Theorem 4.1, we prove that $x_n \to 0$ as $n \to \infty$.

4.2 Discussion of the presence of a small parameter

First of all it is necessary to note that the form of the equation (20), where we have small parameter h by f and $\sqrt{h}$ by S_n, is imposed by the Euler–Maruyama discretization of Itô stochastic differential equation with mesh size h. However, apart of this, in order to guarantee local stability for difference equation, the presence of small parameter by S_n is essential. The following example gives reason for that.

Example 4.5. Let $S_0 = 2$ and $S_n \in (0,\ 0.3)$ for all $n > 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a solution to equation

$$x_{n+1} = x_n - x_n^3 + S_n, \quad n = 1, 2, \ldots,$$

with initial value $x_0 \in (-0.5,\ 0.5)$. Indeed, $x_1 \geq S_0 - |x_0 - x_0^3| \geq 2 - 0.5 = 1.5$ and $|x_{n+1}| \geq |x_n - x_n^3| - S_n \geq 1.875 - 0.3 > 1.5$ for all $n \geq 1$.

4.3 Proof of Theorem 4.1

In order to prove Theorem 4.1, we first establish the boundedness of solutions under the condition (23).

LEMMA 4.6. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ obeys* (2), (22), (23) *and $|S_n| \leq \bar{S}$ for all $n \in \mathbb{N}$. Then there is an $h_0 > 0$ such that for every $h \leq h_0$ there exist $L(h)$, $R(h) > 0$ such that solution $(x_n)_{n\in\mathbb{N}}$ of* (20) *obeys the property: $x_n \in [-L(h),\ R(h)]$ for all $n \in \mathbb{N}$ and when initial value $x_0 \in [-L(h), R(h)]$.*

Proof. As it was mentioned in the beginning of this section, we can assume that (21) holds. We let $a = \bar{S}$ in condition (23) and find $L* > 0$ such that for all $u \geq L*$ we have

$$\sqrt{\frac{u}{f(u)}} \inf_{s\in(0,a\sqrt{u/f(u)})} |f(u-s)| > \bar{S} \quad \text{and} \quad \sqrt{uf(u)} > 2\bar{S}, \tag{29}$$

and for all $u \leq -L*$ we have

$$\sqrt{\frac{u}{f(u)}} \inf_{s \in (0, a\sqrt{u/f(u)})} |f(u+s)| > \bar{S} \quad \text{and} \quad \sqrt{uf(u)} > 2\bar{S}. \tag{30}$$

Let $L_0 \in \mathbb{R}$ be such that $|L_0| \geq L*$ and

$$\frac{f(L_0)}{L_0} = \sup_{|L| \leq |L_0|} \frac{f(L)}{L}. \tag{31}$$

Assume for simplicity that $L_0 > 0$. We define

$$h_0 = \frac{L_0}{f(L_0)}. \tag{32}$$

We fix some $h < h_0$ and define

$$R = R(h) = \inf\left\{u > L_0 : h\frac{f(u)}{u} = 1\right\}, \tag{33}$$

$$L = L(h) = -\inf\left\{u < -L_0 : h\frac{f(u)}{u} = 1\right\}, \tag{34}$$

$$\varepsilon_r = \varepsilon_r(h) = \bar{S}\sqrt{\frac{R}{f(R)}}, \quad \varepsilon_l = \varepsilon_l(h) = \bar{S}\sqrt{\frac{-L}{f(-L)}}. \tag{35}$$

The second relations in (29)–(30) and also (35), imply that

$$\varepsilon_l < L(h) - \varepsilon_l, \quad \varepsilon_r < R(h) - \varepsilon_r.$$

Relations (31)–(35) imply that for all $n \in \mathbb{N}$, $h \leq h_0$ and $u \in [-L(h), R(h)]$

$$\sqrt{h}S_n < \max\{\varepsilon_l, \varepsilon_r\} \quad \text{and} \quad h\frac{f(u)}{u} < 1. \tag{36}$$

Then for $x \in [-L(h), R(h)]$

$$|x - hf(x)| = \left|x\left(1 - h\frac{f(x)}{x}\right)\right| \leq |x|. \tag{37}$$

Applying (29) and (35) we get for $x \in (R - \varepsilon_r, R)$

$$|f(x)| \geq \inf_{s \in (0, \varepsilon_r)} |f(R-s)| \geq \bar{S}\sqrt{\frac{f(R)}{R}},$$

and therefore,

$$h|f(x)| = \frac{R}{f(R)}|f(x)| \geq \frac{R}{f(R)}\bar{S}\sqrt{\frac{f(R)}{R}} = \bar{S}\sqrt{\frac{R}{f(R)}} = \varepsilon_r. \tag{38}$$

Applying (30) and (35) we get for $x \in (-L, -L+\varepsilon_l)$

$$|f(x)| \ge \inf_{s\in(0,\varepsilon_l)} |f(-L+s)| \ge \bar{S}\sqrt{\frac{f(-L)}{-L}},$$

and therefore,

$$h|f(x)| = \frac{-L}{f(-L)}|f(x)| \ge \frac{-L}{f(-L)}\bar{S}\sqrt{\frac{f(-L)}{-L}} = \bar{S}\sqrt{\frac{-L}{f(-L)}} = \varepsilon_l. \tag{39}$$

All the above imply that for all $n \ge N$ and $h \le h_0$

$$\begin{aligned} &|x_{n+1}| < |x_n| + \varepsilon_r, \quad \text{if } x_n \in [0, R(h) - \varepsilon_r],\\ &|x_{n+1}| < |x_n| + \varepsilon_l, \quad \text{if } x_n \in [-L(h) + \varepsilon_l, 0],\\ &|x_{n+1}| \le |x_n|, \quad \text{if } x_n \in [-L(h), -L(h)+\varepsilon_l] \cap [R(h)-\varepsilon_r, R(h)]. \end{aligned} \tag{40}$$

Indeed, the first two lines in (40) obviously follow from (35) and (37). To show the third line we estimate for $x_n \in [R - \varepsilon_r, R]$: when $x_{n+1} > 0$, we get from (38)

$$|x_{n+1}| \le |x_n| - h|f(x_n)| + \varepsilon_r \le |x_n| - \varepsilon_r + \varepsilon_r \le |x_n|,$$

while when $x_{n+1} < 0$, we get from (36)

$$|x_{n+1}| \le -|x_n| + h|x_n|\frac{f(x_n)}{x_n} + \varepsilon_r \le |x_n| - |x_n| + \varepsilon_r = \varepsilon_r < R(h) - \varepsilon_r \le |x_n|.$$

Now we make an estimate in the case $x_n \in [-L(h), L(h)+\varepsilon_l]$: when $x_{n+1} > 0$ we get from (36)

$$|x_{n+1}| \le -|x_n| + h|f(x_n)| + \varepsilon_l \le |x_n| - |x_n| + \varepsilon_l = \varepsilon_l < L(h) - \varepsilon_l \le |x_n|,$$

while when $x_{n+1} < 0$, we get from (39)

$$|x_{n+1}| \le |x_n| - h|x_n|\frac{f(x_n)}{x_n} + \varepsilon_l \le |x_n| - \varepsilon_l + \varepsilon_l \le |x_n|.$$

The relations (40) mean that if $x_0 \in [-L(h), R(h)]$ for all $n \in \mathbb{N}$, we have $x_n \in [-L(h), R(h)]$. □

The proof of Theorem 4.1 now follows rapidly from the results of Lemmas 4.6 and 3.1.

Proof of Theorem 4.1. We fix $h \le h_0$ and define $L(h)$ and $R(h)$. We put $f_h(u) = hf(u)$ and show that assumptions of Lemma 3.1 are fulfilled for $f_h(u)$ when $u \in [-L(h), R(h)]$. Indeed, (3) reduces to

$$\inf_{\{u: -L(h) < u < -c,\, c < u < R(h)\}} |f(u)| > 0$$

and follows from (2), condition (36) implies (4) with $\gamma = 1$. Since Lemma 4.6 guarantees that $x_n \in [-L(h), R(h)]$ for all $n \in \mathbb{N}$, when $x_0 \in [-L(h), R(h)]$, our result follows from Lemma 3.1. □

5. Stability for stochastic difference equations

In this section and the next section, we list and prove the main results of the paper. These concern the stability (both local and global, and both almost sure and with positive probability less than unity) and the instability of solutions of stochastic difference equations (again both local and global and either almost sure or with positive probability less than unity). In this section, we concentrate on the equation (1) which does not contain a parameter $h > 0$ whose value can be made arbitrarily small. In the next section, our analysis tackles the equation (11) with a small parameter, and results which are peculiar to that equation are presented there. In this section, our first subsection deals with results on global stability and instability with probability one. In the second subsection, we show that instability can occur with arbitrary probability, when the initial condition is sufficiently far from the equilibrium.

5.1 Global stability and instability

We notice that $\sigma_n \xi_{n+1} \to 0$ a.s. as $n \to \infty$ is a necessary condition for asymptotic stability, and that in turn it can be expressed in terms of a condition on a summation of probabilities.

Lemma 5.1. *Let Assumption* 2.1 *hold.*

(a) *If*

$$\sum_{n=1}^{\infty} \left[1 - F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right) + F_{n+1}\left(\frac{-\varepsilon}{|\sigma_n|}\right) \right] < \infty, \quad \text{for all } \varepsilon \in \mathbb{R}^+, \tag{41}$$

then $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$, *a.s.*

(b) *If*

$$\sum_{n=1}^{\infty} \left[1 - F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right) + F_{n+1}\left(\frac{-\varepsilon}{|\sigma_n|}\right) \right] = \infty, \quad \text{for some } \varepsilon \in \mathbb{R}^+, \tag{42}$$

then $\limsup_{n\to\infty} |\sigma_n \xi_{n+1}| \geq \varepsilon$, *a.s.*

The proof is an easy consequence of the Borel–Cantelli lemma, and is not given.

Theorem 5.2. *Let Assumption* 2.1 *hold. Suppose f is continuous. Let* $(X_n)_{n\in\mathbb{N}}$ *be a solution to equation* (1) *with arbitrary initial condition* $X_0 \in \mathbb{R}$.

(a) *If the distribution functions* F_n, $n = 1, 2, \ldots$ *obey* (42) *then*

$$\mathbb{P}\left[\lim_{n\to\infty} X_n = 0 \right] = 0.$$

(b) *If the distribution functions* F_n, $n = 1, 2, \ldots,$ *obey*

$$\sum_{n=1}^{\infty} \left[1 - F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right) + F_{n+1}\left(\frac{-\varepsilon}{|\sigma_n|}\right) \right] = \infty, \quad \text{for all } \varepsilon \in \mathbb{R}^+, \tag{43}$$

then $\limsup_{n\to\infty} |X_n| = \infty$, *a.s.*

Proof. To prove part (a), by Lemma 5.1, (42) implies that $\limsup_{n\to\infty}|\sigma_n\xi_{n+1}| > \varepsilon$, a.s. Therefore, if $X_n \to 0$ with positive probability, then $\sigma_n\xi_{n+1} \to 0$ with positive probability, a contradiction. To establish (b), by the Borel–Cantelli lemma and (43), $\limsup_{n\to\infty}|\sigma_n\xi_{n+1}| = \infty$ a.s. Suppose there is an event $A = \{\omega\colon \limsup_{n\to\infty}|X_n(\omega)| < \infty\}$. Now $\sigma_n\xi_{n+1} = X_{n+1} - X_n - f(X_n)$, so as f is continuous, a.s. on A, it must hold that $\limsup_{n\to\infty}|\sigma_n\xi_{n+1}| < \infty$. But, this contradicts an inference of the assumption. Hence, $\limsup_{n\to\infty}|X_n| = \infty$, a.s. □

We comment on some consequences of Theorem 5.2. It is interesting to note that we can have instability if the noise intensity fades to zero as time tends to infinity, once the tails of the distributions of ξ are not compactly supported. Therefore, we see that it is an important problem to determine a critical rate at which the noise intensity fades, so that stability arises for a given type of tail behaviour of the distributions. Later in the paper, we will be able to determine easily verifiable conditions on the rate of decay of σ_n in the case when ξ_n are asymptotically identically distributed as $n \to \infty$.

Moreover, Theorem 5.2 tells us that if the noise has too great an effect (in the sense that (42) or (43) hold), then it does not matter how strong the restoring force f might be, either close to, or far from equilibrium. If too strong a noise perturbation is present, solutions of (1) cannot be stabilised for *any* choice of f. The principal reason for this is due to the noise perturbation being independent of the state of the system.

Armed with this result, we can determine necessary and sufficient conditions for the almost sure asymptotic stability of solutions of (1) under the assumptions (2)–(4) on f.

THEOREM 5.3. *Let Assumption* 2.1 *hold. Let* $f: \mathbb{R} \to \mathbb{R}$ *be continuous and obey* (2)–(4). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a solution to equation* (1) *with arbitrary initial condition* $X_0 \in \mathbb{R}$. *Then the following are equivalent:*

(a) *the distributions* F_n, *and the numbers* σ_n, *for* $n = 1, 2, \ldots,$ *obey* (41);
(b) $\lim_{n\to 0}\sigma_n\xi_{n+1} = 0$, *a.s.*;
(c) $\lim_{n\to\infty}X_n = 0$ *with positive probability*;
(d) $\lim_{n\to\infty}X_n = 0$, *a.s.*

Before giving the proof, we note once again that the conditions for stability for f and the noise are in some sense independent: given the conditions (2)–(4) on f (which do not involve F_n or σ_n), the necessary and sufficient conditions on F_n and σ_n which guarantee stability do not depend on f. Therefore, it does not matter how strong or weak the mean-reverting force of the underlying unperturbed deterministic system may be, the same conditions are required on the noise (viz., on σ and the tails of ξ) to produce stability. Of course, we may expect that the solution may tend to zero more slowly if the noise intensity fades away more slowly, or if the tails of the F_n's are fatter; moreover, the solution may overshoot the equilibrium due to a noise-induced oscillation. See for example Ref. [1] for results on the rate of decay. However, consideration of convergence rates or the presence of oscillation are not our concern in this paper.

We also notice that the conditions in this theorem allow for a wide variety of behaviour of f close to the equilibrium (which must of course be consistent with stability). In particular, there is no requirement that $f(x)$ must have leading order linear term as $x \to 0$; Theorem 5.3 applies equally well to a non-hyperbolic equilibrium as it does to a hyperbolic equilibrium.

Proof of Theorem 5.3. To prove the equivalence, we show that (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (d) $\Rightarrow$ (c).

On any event A' of positive probability on which $X_n \to 0$, due to continuity of f, as $n \to \infty$,

$$\sigma_n \xi_{n+1} = X_{n+1} - (X_n - f(X_n)) \to 0, \quad \text{a.s. on } A'.$$

Therefore, $\mathbb{P}[\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0] > 0$. Thus, as the ξ's are independent, by Kolmogorov's zero-one law, it follows that $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$ a.s. Thus, (c) implies (b). (b) implies (a) by Lemma 5.1. To prove (a) implies (d), we note that for almost all $\omega \in \Omega$ equation (1) can be considered as a deterministic equation of type (14) with $S_n = \sigma_n \xi_{n+1}(\omega)$. Since (a) implies $\sigma_n \xi_{n+1}(\omega) \to 0$ for almost all $\omega \in \Omega$ and the function f obeys the conditions (2)–(4), we can apply Lemma 3.1. As a result we find for almost all $\omega \in \Omega$ that $X_n(\omega) = x_n \to 0$, as $n \to \infty$, proving (d). Since (d) evidently implies (c) the proof is complete. □

To prove our instability result, we must show that on an event of arbitrary probability less than unity, a uniform bound can be placed on $\sigma_n \xi_{n+1}$ which, on that event, depends only on the probability of the event.

Lemma 5.4. *Let* (41) *hold. Then for all* $\gamma \in (0, 1)$ *there exist* $\Omega_\gamma \subseteq \Omega$ *and* $j(\gamma) > 0$ *such that*

$$\max_{n\in\mathbb{N}} |\sigma_n \xi_{n+1}(\omega)| < j(\gamma), \quad \omega \in \Omega_\gamma, \ \ \mathbb{P}[\Omega_\gamma] > 1 - \gamma. \tag{44}$$

Lemma 5.4 holds true if $\sigma_n \xi_{n+1}$ is only bounded, i.e. there is a non random $C > 0$ and $N(C, \omega)$ such that $|\sigma_n \xi_{n+1}| \leq C$ for all $n \geq N(C, \omega)$ a.s.

Before stating the instability result, we pause to note that Lemma 5.4 is existential in character; it does not give a constructive estimate of the size of the bound $j(\gamma)$. An interesting and practical question, which is not addressed here is the following: can we construct such a uniform bound on $\sigma_n \xi_{n+1}$ on a set of arbitrary probability which depends on properties of the distributions F_n and the sequence $(\sigma_n)_{n\in\mathbb{N}}$? A satisfactory resolution of this question would give an *a priori* estimate of the stability basin of zero; in control engineering or in economics, the determination of such an explicit stability basin is of the first order of importance.

Theorem 5.5. *Let conditions* (18) *hold. Let* (41) *hold. Let* $(X_n)_{n\in\mathbb{N}}$ *be a solution of* (1) *with initial condition* $X_0 \in \mathbb{R}$. *Then for all* $\gamma \in (0, 1)$ *there exist* $\Omega_\gamma \subseteq \Omega$, $\mathbb{P}(\Omega_\gamma) > 1 - \gamma$, *and* $d(\gamma) > 0$, *such that for all* ς, $|\varsigma| > d(\gamma)$, *we have for* $\omega \in \Omega_\gamma$, *a.s,*

$$\liminf_{n\to\infty} X_n(\omega) = -\infty, \quad \limsup_{n\to\infty} X_n(\omega) = \infty.$$

For the proof, we apply Lemmas 5.4 and 3.4.

Theorem 5.5 does not tell us whether it is possible to have stability with positive probability (or even with probability one) with *sufficiently small* initial condition. In the next section, however, local stability results are presented.

6. Stability of stochastic difference equations with a small parameter

In this section, we state and prove results which hold for the local stability of the solution of a stochastic difference equation with small parameter $h > 0$, namely:

$$X_{n+1} = X_n - hf(X_n) + \sqrt{h}\sigma_n \xi_{n+1}, \quad n = 1, 2, \ldots, \quad X_0 \in \mathbb{R}. \tag{45}$$

We also comment on the connection between the stability of solutions of (45) and the stability of a related Itô stochastic differential equation, and show under appropriate conditions that the almost sure asymptotic behaviour of the discretised equation is consistent with that of the original Itô equation.

6.1 General stability and instability results

Theorems 5.2, 5.3 and 5.5 can be applied to (45) directly; we leave the formulation of such results for equation (45) to the reader. However, there is an additional result which can be stated for the local stability of solutions of (45) in which the presence of a small parameter is necessary.

THEOREM 6.1. *Suppose the continuous function* $f : \mathbb{R} \to \mathbb{R}$ *obeys* (2), (22), (23) *and let* σ *obey* (41). *Then for any* $\gamma \in (0, 1)$ *there is a number* $h(\gamma) > 0$ *and an event* $\Omega_\gamma \subseteq \Omega$ *with* $\mathbb{P}[\Omega_\gamma] > 1 - \gamma$, *such that for every* $h \le h(\gamma)$ *there exist* $L(\gamma, h)$, $R(\gamma, h) > 0$ *such that when* $X_0 \in [-L(\gamma, h), \; R(\gamma, h)]$, *the solution* $(X_n)_{n \in \mathbb{N}}$ *of* (14) *obeys the property:* $\lim_{n\to\infty} X_n(\omega) = 0$ *for* $\omega \in \Omega_\gamma$ *a.s.*

Proof. By Lemma 5.4, we find Ω_γ and $j(\gamma)$. Proceeding as in Lemma 4.6, for $\bar{S} = j(\gamma)$ we find $L^* = L^*(\gamma)$, $L_0 = L_0(\gamma)$ and $h_0 = h_0(\gamma)$ (by (32)). We fix some $h < h_0(\gamma)$ and define $R(\gamma, h)$ and $L(\gamma, h)$ by the formulae (33) and (34), respectively. Then Lemma 4.6 implies that $X_n(\omega) \in [-L(\gamma, h), \; L(\gamma, h)]$ when $X_0 \in [-L(\gamma, h), \; L(\gamma, h)]$ and $\omega \in \Omega_\gamma$. We complete the proof by applying Theorem 4.1. □

Before proceeding further, we pause to examine one aspect of this result and its proof. The proof constracts the interval $[-L, R]$ from the properties of f, the size of h, and the uniform bound $j(\gamma)$ on $|\sigma_n \xi_{n+1}|$ which holds on Ω_γ. In fact, the length of the interval increases with decreasing h and decreases with increasing γ. From the perspective of numerical simulation it is apparent that the basin of attraction increases in extent if the mesh size h decreases, suggesting that it might be possible to show, under the conditions (2), (22), and (23), that there is an arbitrarily high probability of convergence on an arbitrarily large interval, provided that there was *a priori* control on the size of $|\sigma_n \xi_{n+1}|$ on a set of appropriately large probability. An investigation of this type is not conducted here.

If this result is taken in conjunction with Theorem 5.2 applied to (45), we see that for highly nonlinear f, solutions with sufficiently large initial condition are unstable with arbitrary probability, while solutions with relatively small initial conditions are stable with arbitrary probability. Therefore, the result of Theorem 5.3, which shows that convergence for all initial conditions is almost sure, is not universal among all stochastic equations of the form (10). In Theorem 5.3, it is the restriction on the linear growth bound on f in (4) which is decisive in yielding a.s. global stability.

6.2 Connection with stochastic differential equations

Consider now the stochastic differential equation

$$X(t) = \varsigma - \int_0^t f(X(s))\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B(s), \quad t \geq 0. \tag{46}$$

Here, B is a standard Brownian motion, f is locally Lipschitz continuous, and σ is continuous. Under the condition (2) on f, it is well-known that there is a unique continuous solution of (46) on $[0, \infty)$. Defining $\sigma_n(h) = \sigma(nh)$, the Euler–Maruyama approximation of (46) on a mesh of uniform length $h > 0$ is given by

$$\hat{X}((n+1)h) = \hat{X}(nh) - hf(\hat{X}(nh)) + \sqrt{h}\sigma_n \xi_{n+1}, \quad n \geq 0; \;\; \hat{X}(0) = \varsigma.$$

If we seek a strong approximation of (46), the random variables $(\xi_n)_{n\in\mathbb{N}}$ are independently and identically normally distributed random variables with zero mean and unit variance. Therefore, putting $X_n = \hat{X}(nh)$, we get the stochastic difference equation

$$X_{n+1}(h) = X_n(h) - hf(X_n(h)) + \sqrt{h}\sigma_n(h)\xi_{n+1}, \quad n \geq 0; \;\; X_0(h) = \varsigma, \tag{47}$$

which is in the form (45).

For simplicity, we consider only a result on the dynamic consistency of equations (46) and (47) when f is globally linearly bounded. This is partly because under the conditions (2) and (3) alone the asymptotic behaviour of X obeying (46) is unknown.

In Theorem 6.1, we obtain asymptotic stability of the solution to the stochastic difference equation where f is not globally linearly bounded. However, for the purpose of practical numerical analysis it is advantageous to obtain a constructive *a priori* estimate on the fluctuations of the noise term. The results in this paper do not provide such an estimate. Therefore, an analysis of this type seems to be a fruitful avenue for future research.

THEOREM 6.2. *Suppose that f is locally Lipschitz function which obeys* (2) *and* (3). *Suppose also that there is a $K > 0$ such that $|f(x)| \leq K|x|$ for all $x \geq 0$. Let $h > 0$ and $(X_n(h))_{n\in\mathbb{N}}$ be the solution of* (47). *Consider the statements*

$$\lim_{t\to\infty} \sigma^2(t)\log t = 0, \tag{48}$$

and

$$\exists h_0 > 0 \quad \text{such that } \; \forall h < h_0, \;\; \lim_{n\to\infty} X_n(h) = 0, \;\; a.s. \tag{49}$$

Then the following assertions are true:

(a) *If* (48) *holds, then* (49) *is true.*
(b) *If $t \mapsto |\sigma(t)|$ is a non-decreasing function and* (49) *holds, then* (48) *is true.*

Therefore, if $t \mapsto |\sigma(t)|$ is a non-decreasing function, (48) *and* (49) *are equivalent.*

This result shows that, for sufficiently small discretisation mesh which can be chosen *a priori*, the asymptotic stability of the discrete problem occurs under appropriate and general conditions on f and σ which also ensure the stability of the continuous equation (46). To justify this assertion, we restate a result of Chan and Williams [4].

THEOREM 6.3. *Suppose that f is an increasing and locally Lipschitz function which obeys* (2) *and satisfies*

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = -\infty.$$

Let X be the unique continuous and $\mathcal{F}^B$-adapted process which obeys (46). Then the following assertions are true:

(a) If (48) holds, then

$$\lim_{t\to\infty} X(t) = 0, \quad \text{a.s.} \tag{50}$$

(b) If $t \mapsto |\sigma(t)|$ *is a non-decreasing function and (50) holds, then (48) is true.*

Therefore, if $t \mapsto |\sigma(t)|$ *is a non-decreasing function,* (48) *and* (50) *are equivalent.*

To consider the connection between the theorems, it is necessary that *f* obeys all conditions in both Theorems. It is an open problem to remove the monotonicity on *f* in Theorem 6.3.

Proof of Theorem 6.2. Let γ be any number in (0, 2) and fix $h_0 = (2-\gamma)/K > 0$. Then for $h < h_0$, the function $f_h(x) = hf(x)$ obeys $|f_h(x)| \le Kh|x| < Kh_0|x| = (2-\gamma)|x|$ for all $x \in \mathbb{R}$. Therefore, by Theorem 5.3, we have for any fixed $h < h_0$ that $X_n(h) \to 0$ as $n \to \infty$ a.s., provided that for that same value of h we have $\lim_{n\to\infty}\sigma_n(h)\xi_{n+1} = 0$ a.s. By (48), we have that $\lim_{n\to\infty}\sigma_n(h)^2 \log n = 0$. In the case when ξ are identically and independently distributed normal random variables, it is seen from Proposition 7.6 and part (iii) of Example 7.9 that this suffices to prove $\lim_{n\to\infty}\sigma_n(h)\xi_{n+1} = 0$ a.s. This proves part (a). To prove part (b), we see from Theorem 5.3 that $X_n(h) \to 0$ a.s. as $n \to \infty$ implies $\sigma_n(h)\xi_{n+1} \to 0$ a.s. as $n \to \infty$. Since $t \mapsto |\sigma(t)|$ is non-increasing, it follows that $n \mapsto |\sigma_n(h)|$ is non-increasing. Therefore, by part (b) of Proposition 7.6 and part (iii) of Example 7.9, we see that $\sigma_n(h)^2 \log(n) \to 0$ as $n \to \infty$. Therefore, for any $h > 0$ we have $\lim_{n\to\infty}\sigma^2(nh)\log(nh) = 0$. Now fix $h > 0$. Then, for every $\varepsilon > 0$ there is $N(\varepsilon) \in \mathbb{N}$ with $N(\varepsilon) > 1$ such that for all $n > N(\varepsilon)$, we have $\sigma^2(nh)\log(nh) < \varepsilon$. Now, let $T(\varepsilon) = (N(\varepsilon)+1)h$. Then for $t > T(\varepsilon)$, there exists $n > N(\varepsilon)$ such that $nh \le t < (n+1)h$, and as $|\sigma|$ is non-increasing, we have

$$\sigma^2(t)\log t \le \sigma^2(nh)\log((n+1)h) < \varepsilon\frac{\log((n+1)h)}{\log(nh)}.$$

Now, as $n \ge 1$, we have $\log(nh) \ge \log h$, and so $2\log(nh) \ge \log((n+1)h)$. Thus for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $\sigma^2(t)\log t < 2\varepsilon$, which proves (48), and with it, part (b). □

7. Asymptotic behaviour of ξ_n and noise term $\sigma_n\xi_{n+1}$

In this section, in order to find the least restrictive conditions on σ_n, which guarantee that $\sigma_n\xi_{n+1} \to 0$, we establish connection between rate of decay of σ_n and asymptotical behaviour of tails of probability distribution functions F_n of random variables ξ_n: $1 - F_n(y)$, as $y \to \infty$ and $F_n(y)$, as $y \to -\infty$.

For the sake of simplicity we assume that distributions of ξ_n are symmetrical, then we can impose restrictions only on the limiting behavior of $1 - F_n(y)$, as $y \to \infty$. We note that this restriction can be significantly relaxed, however, the purpose of this paper is not to consider general tails distribution rather then to show that our results hold for the most important cases of distributions of ξ_n.

ASSUMPTION 7.1. Suppose that $(\xi_n)_{n\in\mathbb{N}}$ are independent, continuously distributed random variables with asymptotically symmetrical distribution functions F_n, i.e. $1-F_n(y)=F_n(-y)$ for all sufficiently large $y>0$, and with $\mathbb{E}\xi_n=0, \mathbb{E}\xi_n^2=1$. Suppose also that supp $F_n=(-\infty,\infty)$. Let $p:[0,\infty)\to[0,\infty)$ *be an increasing continuous function and*

$$[1-F_n(y)]e^{p(y)}\to \text{constant}\neq 0, \quad \textit{as } y\to\infty, \ \textit{uniformly in } n\in\mathbb{N}. \tag{51}$$

Since p is strictly increasing, it has an inverse function, $p^{-1}(y)$. We define

$$q_n=p^{-1}(\log(n-1)). \tag{52}$$

Define r and r_n by

$$r(x)=p^{-1}(\log x), \tag{53}$$

$$r_n=r(n), \tag{54}$$

and note that $r_n\uparrow\infty$ as $n\to\infty$.

In this section, by proving an intermediate result about the rate of growth of the almost sure partial maxima of the sequence ξ_n, we derive conditions on σ_n which ensue that $\sigma_n\xi_{n+1}\to 0$ a.s. As is seen in the following subsection, these conditions are also close to being necessary. Necessary and sufficient results for the convergence of $\sigma_n\xi_{n+1}\to 0$ are stated later in this section under specific stronger conditions on the tails of ξ and the noise intensity σ.

7.1 Conditions with general tails

PROPOSITION 7.2. *Suppose Assumption* 7.1 *holds. Let* r_n *be defined in* (54).

(a) *If*

$$\limsup_{n\to\infty}\frac{\log|\sigma_n|}{\log r_n}<-1 \tag{55}$$

then

$$\lim_{n\to\infty}\sigma_n\xi_{n+1}=0, \quad \text{a.s.} \tag{56}$$

(b) *If* (56) *holds, then*

$$\liminf_{n\to\infty}\frac{\log|\sigma_n|}{\log r_n}\leq -1. \tag{57}$$

(c) *If* $\lim_{n\to\infty}(\log|\sigma_n|/\log r_n)$ *exists, then* (55) *implies* (56), *and* (56) *implies* (57).

A key intermediate result in the proof of Proposition 7.2 is to show that

$$\limsup_{n\to\infty}\frac{\log|\xi_{n+1}|}{\log r_n}=1. \tag{58}$$

The first step to proving this is to establish the following result.

LEMMA 7.3. *Suppose Assumption* 7.1 *holds. Let* $\varepsilon\in(0,1)$. *Define*

$$q_n^{(\varepsilon)}=p^{-1}((1+\varepsilon)\log(n-1)), \tag{59}$$

$$q_n^{(-\varepsilon)}=p^{-1}((1-\varepsilon)\log(n-1)). \tag{60}$$

Then for every $\varepsilon\in(0,1)$ *there exist a.s. events* Ω_ε^+ *and* Ω_ε^- *such that*

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n^{(\varepsilon)}}\leq 1,\quad \text{a.s. } on\ \Omega_\varepsilon^+, \tag{61}$$

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n^{(-\varepsilon)}}\geq 1,\quad \text{a.s. } on\ \Omega_\varepsilon^-. \tag{62}$$

We assume

$$\limsup_{x\to\infty}\frac{\log r(x^{1+\varepsilon})}{\log r(x)}\leq C_\varepsilon, \tag{63}$$

where

$$\lim_{\varepsilon\to 0}C_\varepsilon=1. \tag{64}$$

LEMMA 7.4. *Let Assumption* 7.1 *hold. Suppose that r defined by* (53) *obeys* (63) *and* (64). *Then* (58) *holds holds true for* r_n, *defined by* (54).

Example 7.5. Let ξ_n be identically distributed. The following examples consider slower than exponential decay in the tails of the distribution. We give the formula for the corresponding q_n (constructed by formula (52)), and the condition on the rate of decay of σ which ensures that (56) holds.

(i) If there exists $m>2$ such that $1-F(y)\sim y^{-m}$ as $y\to\infty$, then $r_n=n^{1/m}$, and

$$\limsup_{n\to\infty}\frac{\log|\sigma_n|}{\log n}<-\frac{1}{m},$$

then (56) holds.

(ii) If there exists $\alpha>0$ such that $1-F(y)\sim e^{-\log^{1+\alpha}y}$ as $y\to\infty$, then $r_n = e^{\log^{1/(1+\alpha)}n}$, and

$$\limsup_{n\to\infty}\frac{\log|\sigma_n|}{\log^{1/(1+\alpha)}n} < -1,$$

then (56) holds.

We note that the first example describes the case of polynomial tails.

7.2 Exponential and super-exponential tails

In this subsection, we give necessary and sufficient conditions for the convergence of $\sigma_n\xi_{n+1} \to 0$ when the tails of F are *super-exponential*. The superexponential property is embodied in the following condition: for every $\varepsilon \in (0, 1)$ there exists a finite C_ε^+ such that

$$\limsup_{x\to\infty}\frac{r(x^{1+\varepsilon})}{r(x)} \leq C_\varepsilon^+, \tag{65}$$

where

$$\lim_{\varepsilon\to 0} C_\varepsilon^+ = 1. \tag{66}$$

Proposition 7.6. *Suppose Assumption* 7.1 *hold. Define* r_n *by* (54)*, and suppose that* r *defined by* (53) *obeys* (65) *and* (66).

(a) *If*

$$\lim_{n\to\infty}\sigma_n r_n = 0, \tag{67}$$

then (56) *holds.*

(b) *If* $n \mapsto |\sigma_n|$ *is non-increasing, and* (56) *holds, then* (67) *holds.*

Therefore, in the case that $n \mapsto |\sigma_n|$ *is non-increasing, we have that* (67) *and* (56) *are equivalent.*

To prove part (b) of Proposition 7.6, the following auxiliary results are needed.

Lemma 7.7. *Suppose Assumption* 7.1 *holds. Then* (56) *implies that* $\lim_{n\to\infty}\sigma_n = 0$.

Lemma 7.8. *Suppose that* $(a(n))_{n\geq 0}$ *is a non-negative and non-increasing sequence such that* $\sum_{n=0}^{\infty} a(n) < \infty$. *Then* $na(n) \to 0$ *as* $n \to \infty$.

The results are elementary, but were proven in a similar context in Appleby, Riedle and Rodkina [3].

Example 7.9. Let ξ_n be identically distributed. The following distributions obey the property (65), and therefore we can determine the critical rate of decay of σ so that $\sigma_n\xi_{n+1} \to 0$ a.s.

(i) There exists $\alpha > 0$ such that $1 - F(y) + F(-y) \sim \mathrm{e}^{-y^\alpha}$, then $r_n = \log^{1/\alpha}n$, and so $\sigma_n\log^{1/\alpha}n \to 0$ as $n \to \infty$ is the critical rate of decay;

(ii) $1 - F(y) + F(-y) \sim e^{-e^y}$, then $r_n = \log\log n$, and so $\sigma_n \log\log n \to 0$ as $n \to \infty$ is the critical rate of decay;

(iii) There is $l > 0$ and $k > 0$ such that $1 - F(y) + F(-y) \sim (c/y)\,e^{-(1/k)y^2}$, then $r_n = \sqrt{k \log n}$, and so $\sigma_n \sqrt{\log n} \to 0$ as $n \to \infty$ is the critical rate of decay.

A standardised normal distribution is covered by case (iii) with $l = 1$, $k = 2$.

7.3 Polynomial tails

We finally show for polynomially decaying tails that a simple summability condition, which does not involve monotonicity, is necessary and sufficient for (56) to hold.

THEOREM 7.10. *Let Assumption* 201 *hold. Suppose that for some* $M \geq 2$

$$[1 - F_n(y)]y^M \to \text{constant} \neq 0, \quad \textit{as } y \to \infty, \ \textit{uniformly in } n \in \mathbb{N}.$$

Then (56) *holds if and only if* $\sum_{i=1}^{\infty} \sigma_i^M < \infty$.

Proof. Applying Borel–Cantelli arguments we can show that $\sigma_n \xi_{n+1} \to 0$, a.s., is equivalent to the following:

$$\sum_{i=1}^{\infty} \left(1 - F_{i+1}\left[\frac{\varepsilon}{\sigma_i}\right]\right) < \infty \quad \text{for all } \varepsilon > 0,$$

which in turn, is equivalent to $\sum_{i=1}^{\infty} \sigma_i^{-M} < \infty$. □

7.4 Stability without square-summable σ

We now improve a result of Ref. [1], where $X_n \to 0$ a.s. was proved in case when $\sum_{n+1}^{\infty} \sigma_n^2 < \infty$.

LEMMA 7.11. *Let Assumption* 7.1 *hold and let* ξ_n *be identically distributed. Suppose there exists* $\varepsilon > 0$ *such that*

$$(1 - F(x) + F(-x))x^{2+\varepsilon} \quad \textit{is bounded}.$$

Then there exists σ_n *such that* $\sum_{n=1}^{\infty} \sigma_n^2 = \infty$, *but solution* $(X_n)_{n \in \mathbb{N}}$ *of equation* (1) *with this particular* σ_n *obeys* $\lim_{n\to\infty} X_n = 0$ *a.s.*

Proof. Since $p(y) > 2(1+\varepsilon)\log y$ for some $y_0 > 0$ and all $y \geq y_0$, we have for all $y \geq y_0$ $p^{-1}(y) < e^{y/(2(1+\varepsilon))}$. Then, for all $n \geq e^{y_0}$

$$r_n = p^{-1}(\log n) < e^{\log n/(2(1+\varepsilon))} = n^{1/(2(1+\varepsilon))}.$$

Let $\delta \in (0, \varepsilon)$ and let $\sigma_n = n^{-((1+\delta)/2(1+\varepsilon))}$. Then for all $n \geq e^{y_0}$

$$\frac{\log \sigma_n}{\log r_n} = -(1+\delta) < -1,$$

i.e. $\limsup_{n\to\infty} \log \sigma_n / \log r_n < -1$. Then, by Lemmas 7.2 and 5.3, $X_n \to 0$, as $n \to \infty$. However,

$$\sum_{n=1}^{\infty} \sigma_n^2 = \sum_{n=1}^{\infty} n^{-(1+\delta/1+\varepsilon)} = \infty.$$

□

Example 7.12. Using Example 7.5, we can conclude the following.

(1) Let $p(y) = 3\log n$. Then $r_n = n^{1/3}$. For $\sigma_n = n^{-2/5}$ both conditions (55) and $\sum_{n+1}^{\infty} \sigma_n^2 = \infty$ hold.
(2) Let $p(y) = \log^{1+\alpha} n$. Then $r_n = \mathrm{e}^{\log^{1/1+\alpha} n}$. For $\sigma_n = \mathrm{e}^{-(1+\varepsilon)\log^{1/1+\alpha} n}$, for all $\varepsilon > 0$, both conditions (55) and $\sum_{n+1}^{\infty} \sigma_n^2 = \infty$ hold.

8. Proofs

8.1 Proof of Lemma 3.2

(i) From (14) we have for all $n \in \mathbb{N}$

$$x_{n+1} - x_n + f(x_n) = S_n. \tag{68}$$

Since $f(0) = 0$ and f is continuous, left hand side of (68) tends to 0, as $n \to \infty$. Then S_n has also tend to zero.

(ii) We suppose that $x_n \to 0$ but there is $x_0^* > 0$ such that $f(x_0^*) = 0$. Then we take x_0^* as an initial value and obtain that

$$x_1 = x_0^* + f\left(x_0^*\right) = x_0^* \Rightarrow x_n = x_0^*, \quad \text{for all } n = 1, 2, \ldots,$$

which contradicts to the fact that $x_n \to 0$.

(iii) Suppose that $f(\bar{u}) = 2\bar{u}$ for some $\bar{u} > 0$. Then by symmetry, $f(-\bar{u}) = -2\bar{u}$. We take $\bar{u}$ as an initial value and obtain that

$$x_1 = \bar{u} - 2\bar{u} = -\bar{u}, \quad x_2 = -\bar{u} + 2\bar{u} = \bar{u},$$

which contradicts to the fact that $x_n \to 0$.

(iv) Let

$$x_n = \sum_{i=1}^{n} S_i,$$

then $x_{n+1} - x_n = S_{n+1}$. Then $x_n \to \infty$, since $\sum_{i=1}^{\infty} S_i = \infty$, and $x_{n+1} - x_n \to 0$. For any $n \in \mathbb{N}$ we define

$$f(x_n) = x_n - x_{n+1} + S_n, \quad f(0) = 0,$$

and note that $f(x_n) = S_n - S_{n+1} > 0$ and $f(x_n) \to 0$. We define $f(u)$ for $u \in (x_n, x_{n+1})$ as a linear segment, connecting points $f(x_n)$ and $f(x_{n+1})$. In the same manner we define $f(u)$ on the interval $(0, x_1)$: as a linear segment, connecting points 0 and $f(x_1)$. We define $f(u)$ for $u < 0$ by $f(u) = -f(-u)$. Then $f(u) \to 0$, if $u \to \infty$.

8.2 Proof of Lemma 4.2

Suppose that $u \geq 0$ and $u - a\sqrt{u/f(u)} > 0$. For $s \in (0, a\sqrt{u/f(u)})$ we have

$$f(u-s) = f(u) - f'(\theta)s$$

with some θ: $u - s \leq \theta \leq u$. By monotonicity of f' we have

$$f(u-s) = f(u) - f'(\theta)s \geq f(u) - f'(u)a\sqrt{\frac{u}{f(u)}}.$$

We estimate

$$\sqrt{\frac{u}{f(u)}} \inf_{s\in(0,a\sqrt{u/f(u)})} f(u-s) \geq \sqrt{\frac{u}{f(u)}} f(u) - f'(u)a\frac{u}{f(u)} = \sqrt{uf(u)} - af'(u)\frac{u}{f(u)}. \quad (69)$$

Substituting condition (27) in (69) we obtain:

$$\sqrt{\frac{u}{f(u)}} \inf_{s\in(0,a\sqrt{u/f(u)})} f(u-s) \geq \sqrt{uf(u)} - aKf^{1+\varepsilon}(u)\frac{u}{f(u)}$$

$$= \sqrt{uf(u)}\left[1 - aK\frac{u^{1/2}}{f^{1/2-\varepsilon}(u)}\right] \to \infty,$$

since, by the fact that

$$\lim_{x\to\infty} \frac{f'(x)}{f(x)^{1+\varepsilon}} = 0,$$

we have $u^{1/2}/f^{1/2-\varepsilon}(u) \to 0$ and also $\sqrt{uf(u)} \to \infty, \ \ u \to \infty$.

Suppose that $u < 0$ and $u - a\sqrt{u/f(u)} < 0$. For $s \in (0, a\sqrt{u/f(u)})$ we have

$$f(u+s) = f(u) + f'(\theta)s$$

with some θ: $u \leq \theta \leq u + s$. By monotonicity of f' we have

$$|f(u+s)| \geq |f(u)| - f'(\theta)s \geq |f(u)| - f'(u)a\sqrt{\frac{u}{f(u)}}.$$

We estimate

$$\sqrt{\frac{u}{f(u)}} \inf_{s\in(0,a\sqrt{u/f(u)})} |f(u+s)| \geq \sqrt{\frac{u}{f(u)}}|f(u)| - f'(u)a\frac{u}{f(u)}$$

$$= \sqrt{uf(u)} - af'(u)\frac{u}{f(u)}. \quad (70)$$

Substituting condition (27) in (70) we obtain:

$$\sqrt{\frac{u}{f(u)}}\inf_{s\in(0,a\sqrt{u/f(u)})}|f(u+s)| \geq \sqrt{uf(u)} - aK|f(u)|^{1+\varepsilon}\frac{u}{f(u)}$$

$$= \sqrt{uf(u)}\left[1 - aK\frac{|u|^{1/2}}{|f(u)|^{1/2-\varepsilon}}\right] \to \infty,$$

since $|u|^{1/2}/|f(u)|^{1/2-\varepsilon} \to 0$, $\sqrt{uf(u)} \to \infty$, $u \to -\infty$.

8.3 Proof of Lemma 5.4

Let $U_n = \sigma_n \xi_{n+1}$. We fix some $\delta_0 \in (0, 1)$ then for all $\omega \in \Omega$ there is $N(\delta_0, \omega)$ such that $|U_n(\omega)| \leq \delta_0$, $n \geq N(\delta_0, \omega)$. For for all $\omega \in \Omega$ we set

$$\theta(\omega) = \max_{i=1,\dots,N(\delta_0,\omega)}\{|U_i(\omega)|\},$$

$$\underline{\Omega}_j = \{\omega : j-1 \leq \theta(\omega) < j\}, \quad \Omega_j = \{\omega : \theta(\omega) < j\} = \bigcup_{i=1}^{j} \underline{\Omega}.$$

Then $\underline{\Omega}_j \cap \underline{\Omega}_i = \emptyset$, when $j \neq i$, and $\Omega = \cup_{j=1}^{\infty}\underline{\Omega}_j$. Therefore, $1 = \mathbb{P}(\Omega) = \sum_{i=1}^{\infty}\mathbb{P}(\underline{\Omega}_i)$, and for every $\gamma \in (0, 1)$ we can find $j(\gamma)$ such that for all $j > j(\gamma)$

$$\mathbb{P}(\Omega_j) = \sum_{i=1}^{j}\mathbb{P}(\underline{\Omega}_i) > 1 - \gamma.$$

We let $\Omega_\gamma = \Omega_{j(\gamma)}$ and observe that $\mathbb{P}(\Omega_\gamma) > 1 - \gamma$ and $\theta(\omega) < j(\gamma)$ when $\omega \in \Omega_\gamma$. Since $\delta_0 < 1$, we also have $\max_{n\in\mathbb{N}}|\sigma_n\xi_{n+1}(\omega)| < j(\gamma)$ a.s. for $\omega \in \Omega_\gamma$.

8.4 Proof of Lemma 7.2

First, we note that $\sigma_n\xi_{n+1} \to 0$ a.s. is equivalent to

$$\log|\sigma_n| + \log|\xi_{n+1}| \to -\infty. \tag{71}$$

Suppose $\limsup_{n\to\infty}\log|\sigma_n|/\log r_n < -1$. Then there exists $\varepsilon > 0$ and $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ we have $\log|\sigma_n| < -(1+\varepsilon)\log r_n$. By condition (58), there exists $N_1(\varepsilon) > N(\varepsilon)$, such that for all $n \geq N(\varepsilon)$ we have $\log|\xi_{n+1}| < (1+\varepsilon/2)\log r_n$. Then

$$\log|\sigma_n| + \log|\xi_{n+1}| \leq -(1+\varepsilon)\log r_n + \left(1+\frac{\varepsilon}{2}\right)\log r_n = -\frac{\varepsilon}{2}\log r_n \to -\infty.$$

Thus (71) and therefore, (6) holds.

Suppose (71) holds, but

$$\liminf_{n\to\infty}\frac{\log|\sigma_n|}{\log r_n} > -1.$$

Then there exists sufficiently small $\varepsilon > 0$ and $N_2(\varepsilon)$ such that for all $n \geq N_2(\varepsilon)$ we have $\log|\sigma_n|/\log\ r_n > -1 + \varepsilon$. By (58) there exists $\{n_k\}$ such that for all $k \in \mathbb{N}$ we get

$\log|\xi_{n_k+1}|/\log r_{n_k} > 1 - \varepsilon/2$. Then

$$\log|\sigma_{n_k}| + \log|\xi_{n_k+1}| \geq (-1+\varepsilon)\log r_{n_k} + \left(1 - \frac{\varepsilon}{2}\right)\log r_{n_k} = \frac{\varepsilon}{2}\log r_{n_k} \to \infty.$$

The contradiction we have thus obtained completes the proof.

8.5 Proof of Lemma 7.3

Since $q_n^{(\varepsilon)} \to \infty$ as $n \to \infty$, and $e^{p(q_n^{(\varepsilon)})} = (n-1)^{1+\varepsilon}$, by Assumption 7.1, we have that $\lim_{n\to\infty}[1 - F_{n+1}(q_n^{(\varepsilon)})](n-1)^{1+\varepsilon}$ exists. By the symmetry of F_{n+1}, we get $\lim_{n\to\infty} F_{n+1}(-q_n^{(\varepsilon)})(n-1)^{1+\varepsilon}$ exists. Hence,

$$\sum_{n=1}^{\infty}[1 - F_{n+1}(q_n^{(\varepsilon)}) + F_{n+1}(-q_n^{(\varepsilon)})] < \infty.$$

By the Borel–Cantelli Lemma, (61) holds. The proof of (62) follows similarly, using $e^{p(q_n^{(-\varepsilon)})} = (n-1)^{1-\varepsilon}$ and in addition employing the independence of $(\xi_n)_{n\in\mathbb{N}}$ and the Borel–Cantelli lemma.

8.6 Proof of Lemma 7.4

Let $\varepsilon \in (0, 1)$ and let $q_n^{(\varepsilon)}$ and $q_n^{(-\varepsilon)}$ be as in Lemma 7.3. By Lemma 7.3, we have

$$\limsup_{n\to\infty}\frac{\log|\xi_n|}{\log q_n} = \limsup_{n\to\infty}\frac{\log|\xi_n|}{\log q_n^{(\varepsilon)}}\cdot\frac{\log q_n^{(\varepsilon)}}{\log q_n} \leq \limsup_{n\to\infty}\frac{\log q_n^{(\varepsilon)}}{\log q_n}.$$

By (63) and the definition of $q_n^{(\varepsilon)}$ and q_n we have

$$\limsup_{n\to\infty}\frac{\log q_n^{(\varepsilon)}}{\log q_n} = \limsup_{n\to\infty}\frac{\log r((n-1)^{1+\varepsilon})}{\log r(n-1)} \leq C_\varepsilon,$$

and so

$$\limsup_{n\to\infty}\frac{\log|\xi_n|}{\log q_n} \leq C_\varepsilon, \quad \text{a.s. on } \Omega_\varepsilon^+.$$

Now consider $\Omega_*^+ = \cap_{\varepsilon\in\mathbb{Q}\cap(0,1)}\Omega_\varepsilon^+$. Then as $C_\varepsilon \to 1$ as $\varepsilon \to 0^+$, we get

$$\limsup_{n\to\infty}\frac{\log|\xi_n|}{\log q_n} \leq 1, \quad \text{a.s. on } \Omega_*^+. \tag{72}$$

Furthermore, $\mathbb{P}[\Omega_*^+] = 1$.

By a similar argument we arrive at

$$\limsup_{n\to\infty}\frac{\log|\xi_n|}{\log q_n} \geq \liminf_{n\to\infty}\frac{\log q_n^{(-\varepsilon)}}{\log q_n} = \left(\limsup_{n\to\infty}\frac{\log q_n}{\log q_n^{(-\varepsilon)}}\right)^{-1}.$$

However, by the definition of $q_n^{(-\varepsilon)}$ and q_n we have

$$\limsup_{n\to\infty} \frac{\log q_n}{\log q_n^{(-\varepsilon)}} = \limsup_{n\to\infty} \frac{\log r(n-1)}{\log r((n-1)^{1-\varepsilon})} \le \limsup_{x\to\infty} \frac{\log r(x)}{\log r(x^{1-\varepsilon})},$$

where the second lim sup is taken through the reals. Next, there is $\varepsilon' > 0$ such that $1 + \varepsilon' = (1-\varepsilon)^{-1}$. Thus (57) gives

$$\limsup_{x\to\infty} \frac{\log r(x)}{\log r(x^{1-\varepsilon})} = \limsup_{y\to\infty} \frac{\log r(y^{1+\varepsilon'})}{\log r(y)} \le C_{\varepsilon'} = C_{\varepsilon(1-\varepsilon)^{-1}}.$$

Inserting this estimate into the previous two inequalities gives

$$\limsup_{n\to\infty} \frac{\log|\xi_n|}{\log q_n} \ge 1/C_{\varepsilon(1-\varepsilon)^{-1}}, \quad \text{a.s. on } \Omega_\varepsilon^-.$$

Since $C_\varepsilon \to 1$ as $\varepsilon \to 0^+$, we obtain

$$\limsup_{n\to\infty} \frac{\log|\xi_n|}{\log q_n} \ge 1, \quad \text{a.s. on } \Omega_*^-, \tag{73}$$

where $\Omega_*^- = \cap_{\varepsilon\in\mathbb{Q}\cap(0,1)}\Omega_\varepsilon^-$, and Ω_*^- has the property that $\mathbb{P}[\Omega_*^-] = 1$. Combining (72) and (73), on the almost sure event $\Omega_* = \Omega_*^+ \cap \Omega_*^-$ we have

$$\limsup_{n\to\infty} \frac{\log|\xi_n|}{\log q_n} = 1, \ \text{ a.s.},$$

as required, as $q_{n+1} = r_n$.

8.7 *Proof of Proposition 7.6, part (a)*

Let $\varepsilon \in (0, 1)$ and let $q_n^{(\varepsilon)}$ and $q_n^{(-\varepsilon)}$ be as in Lemma 7.3. By Lemma 7.3, we have

$$\limsup_{n\to\infty} \frac{|\xi_n|}{q_n} = \limsup_{n\to\infty} \frac{|\xi_n|}{q_n^{(\varepsilon)}} \cdot \frac{q_n^{(\varepsilon)}}{q_n} \le \limsup_{n\to\infty} \frac{q_n^{(\varepsilon)}}{q_n}.$$

By (65) and the definition of $q_n^{(\varepsilon)}$ and q_n we have

$$\limsup_{n\to\infty} \frac{q_n^{(\varepsilon)}}{q_n} = \limsup_{n\to\infty} \frac{p^{-1}((1+\varepsilon)\log(n-1))}{p^{-1}(\log(n-1))} \le C_\varepsilon^+,$$

and so

$$\limsup_{n\to\infty} \frac{|\xi_n|}{q_n} \le C_\varepsilon^+, \quad \text{a.s. on } \Omega_\varepsilon^+.$$

Now consider $\Omega_*^+ = \cap_{\varepsilon\in\mathbb{Q}\cap(0,1)}\Omega_\varepsilon^+$. Then as $C_\varepsilon^+ \to 1$ as $\varepsilon \to 0^+$, we get

$$\limsup_{n\to\infty} \frac{|\xi_n|}{q_n} \le 1, \quad \text{a.s. on } \Omega_*^+. \tag{74}$$

Furthermore, $\mathbb{P}[\Omega_*^+] = 1$.

By a similar argument we arrive at

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n}\geq \liminf_{n\to\infty}\frac{q_n^{(-\varepsilon)}}{q_n}=\left(\limsup_{n\to\infty}\frac{q_n}{q_n^{(-\varepsilon)}}\right)^{-1}.$$

However, by the definition of $q_n^{(-\varepsilon)}$ and q_n we have

$$\limsup_{n\to\infty}\frac{q_n}{q_n^{(-\varepsilon)}}=\limsup_{n\to\infty}\frac{p^{-1}(\log(n-1))}{p^{-1}((1-\varepsilon)\log(n-1))}\leq \limsup_{x\to\infty}\frac{p^{-1}(\log x)}{p^{-1}((1-\varepsilon)\log x)},$$

where the second lim sup is taken through the reals. Next, there is $\varepsilon'>0$ such that $1+\varepsilon'=(1-\varepsilon)^{-1}$. Thus (65) gives

$$\limsup_{x\to\infty}\frac{p^{-1}(\log x)}{p^{-1}((1-\varepsilon)\log x)}=\limsup_{y\to\infty}\frac{p^{-1}((1+\varepsilon')\log y)}{p^{-1}(\log y)}\leq C^+_{\varepsilon'}=C^+_{\varepsilon(1-\varepsilon)^{-1}}.$$

Inserting this estimate into the previous two inequalities gives

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n}\geq 1/C^+_{\varepsilon(1-\varepsilon)^{-1}},\quad \text{a.s. on } \Omega^-_\varepsilon.$$

Since $C^+_\varepsilon\to 1$ as $\varepsilon\to 0^+$, we obtain

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n}\geq 1,\quad \text{a.s. on } \Omega^-_*, \tag{75}$$

where $\Omega^-_*=\cap_{\varepsilon\in\mathbb{Q}\cap(0,1)}\Omega^-_\varepsilon$, and Ω^-_* has the property that $\mathbb{P}[\Omega^-_*]=1$. Thus on the almost sure event $\Omega_*=\Omega^+_*\cap\Omega^-_*$ we have

$$\limsup_{n\to\infty}\frac{|\xi_n|}{q_n}=1,\quad \text{a.s.}$$

Now, $\sigma_n r_n=\sigma_n q_{n+1}$, so $\sigma_n q_{n+1}\to 0$ as $n\to\infty$. Hence,

$$\lim_{n\to\infty}|\sigma_n\xi_{n+1}|=\lim_{n\to\infty}\sigma_n q_{n+1}\frac{|\xi_{n+1}|}{q_{n+1}}=0,\quad \text{a.s.},$$

completing the proof.

8.8 Proof of Proposition 7.6, part (b)

By Assumption 7.1 we have, uniformly in $n\in\mathbb{N}$,

$$\lim_{x\to-\infty}F_n(x)e^{p(|x|)}=L,\quad \lim_{x\to\infty}[1-F_n(x)]e^{p(x)}=L>0.$$

Since (56), by Lemma 7.7 that we must have $\sigma_n\to 0$ as $n\to\infty$. Now, let ε be a fixed positive rational number. Therefore, the limits

$$\lim_{n\to-\infty}F_{n+1}\left(\frac{-\varepsilon}{|\sigma_n|}\right)e^{p(\varepsilon/\sigma_n)}=L,\quad \lim_{n\to\infty}\left(1-F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right)\right)e^{p(\varepsilon/\sigma_n)}=L,$$

hold and so

$$\lim_{n\to\infty}\left[1 - F_{n+1}\left(\frac{\varepsilon}{|\sigma_n|}\right) + F_{n+1}\left(\frac{-\varepsilon}{|\sigma_n|}\right)\right]\mathrm{e}^{p(\varepsilon/\sigma_n)} = 2L. \tag{76}$$

Now, because $\sigma_n\xi_{n+1} \to 0$ a.s. is equivalent to (41), it follows that

$$\sum_{n=1}^{\infty} \mathrm{e}^{-p(\varepsilon/\sigma_n)} < \infty, \quad \text{for all } \varepsilon \in \mathbb{R}^+. \tag{77}$$

Now, fix $\varepsilon \in \mathbb{R}^+$. Since $n \mapsto |\sigma_n|$ is non-increasing, the sequence $(a_\varepsilon(n))_{n\geq 0}$ defined by $a_\varepsilon(n) = \exp(-p(\varepsilon/|\sigma_n|))$ for $n > N_1(\varepsilon)$ is non-increasing. By Lemma 7.8, $\lim_{n\to\infty} na_\varepsilon(n) = 0$, or $\lim_{n\to\infty} n\exp(-p(\varepsilon/|\sigma_n|)) = 0$. Thus

$$\lim_{n\to\infty}[\log n - p(\varepsilon/|\sigma_n|)] = -\infty.$$

By (54), we have $r_n = p^{-1}(\log n)$, so $p(r_n) = \log n$, hence, $\lim_{n\to\infty}[p(r_n) - p(\varepsilon/|\sigma_n|)] = -\infty$. Thus, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that for all $n > N_2(\varepsilon)$ we have $p(r_n) - p(\varepsilon/|\sigma_n|) < 0$. Since p is non-decreasing, it follows that $|\sigma_n| r_n < \varepsilon$ for all $n > N_2(\varepsilon)$. However, this inequality holds for every $\varepsilon \in \mathbb{R}^+$, and so it follows that $\lim_{n\to\infty}\sigma_n r_n = 0$, as required.

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