Almost Sure Convergence of Solutions to Non-Homogeneous Stochastic Difference Equation

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Abstract

We consider a non-homogeneous nonlinear stochastic difference equation

$$X_{n+1} = X_n\Big(1 + f(X_n)\xi_{n+1}\Big) + S_n, \quad n = 0, 1, \dots,$$

and its linear counterpart

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big) + S_n, \quad n = 0, 1, \dots,$$

both with initial value X_0, non-random decaying free coefficient S_n and independent random variables ξ_n. We establish results on a.s. convergence of solutions X_n to zero. Obtained necessary conditions tie together certain moments of the noise ξ_n and the rate of decay of S_n. To ascertain sharpness of our conditions we discuss some situations when X_n diverges. We also establish a result concerning the rate of decay of X_n to zero.

Several examples are given to illustrate the ideas of the paper.

*Keywords:*Nonlinear stochastic difference equations, almost sure stability, martingale convergence theorem.

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1 Introduction

The theory of stochastic difference equations is relatively young, especially in its nonlinear part. Linear stochastic difference equations with independent identically distributed perturbations (i.i.d.) are the most studied ones (cf [9]) but even for this type of equation there still exist some open questions [20]. In this paper we are going to give answers to some of them and then proceed to discuss a class of nonlinear stochastic difference equations for which very few results are available [12, 13, 19, 21, 22, 23].

The interest towards stochastic difference equations has been on the increase due to their numerous applications and the fact that they serve for numerical simulations of stochastic differential equations (cf [7, 8, 11, 16]). Stability of solutions of stochastic difference equations is also very important in, to give some examples, mathematical finance (asset price evolution in discrete (B, S)-markets) and mathematical biology (population dynamics), see, for example, [6] and references therein.

The main objects of our consideration are the following equations: the non-homogeneous nonlinear stochastic difference equation

$$X_{n+1} = X_n\Big(1 + f(X_n)\xi_{n+1}\Big) + S_n, \qquad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1}$$

and its linear counterpart

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big) + S_n, \qquad n \in \mathbb{N}_0, \tag{2}$$

with initial value $X_0 > 0$, non-random free coefficient S_n and independent random variables ξ_n. Unless explicitly indicated, we do not demand that ξ_n be identically distributed. Everywhere in the paper we suppose that

$$\begin{aligned} &f : \mathbb{R}^1 \to [0, 1] \text{ is continuous and } f(u) = 0 \Leftrightarrow u = 0, \\ &1 + \xi_{n+1} > 0 \quad \text{and} \quad S_n > 0 \qquad \forall\, n \in \mathbb{N}_0. \end{aligned} \tag{3}$$

These conditions guarantee that X_n remains positive for all n.

Equations of the type (1) and (2) are sufficiently complex to require more powerful methods than those used to study, for example, the linear homogenous equation

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big), \qquad n \in \mathbb{N}_0. \tag{4}$$

On the other hand, equations (1) and (2) are sufficiently simple to allow a rather complete understanding of their behaviour. In our paper we use an adaptation of a martingale convergence theorem to prove most of the results. The methods of proof that we develop can also be used on more complicated recursions or in more applied contexts, for example to study the faithfulness properties[1] of numerical solutions to stochastic differential equations.

To get the flavour of our results it is instructive to start with the behaviour of the corresponding deterministic equation,

$$x_{n+1} = x_n\Big(1 + a_{n+1}\Big) + S_n,$$

with $1 + a_{n+1} > 0$ (the nonlinear deterministic equation is discussed in Section 6.1). If $a_n \equiv a$, the solutions converge to zero when $a < 0$ (or $\ln(1+a) < 0$, which is the same) and $S_n \to 0$.

[1] such as the A-stability, which was studied in [7] on the example of the recursion of the type (4). A method is said to be *A-stable* if it correctly predicts the asymptotic stability of the approximated equation.

Now take $S_n \equiv 0$ and allow a_n to contain noise, $a_n = a+\zeta_n$ with $\mathbf{E}\zeta_n = 0$. It is easy to see that the solutions will still tend to zero if $a = \mathbf{E}(a+\zeta_n) < 0$. But they will also tend to zero if $a > 0$ but $\mathbf{E}\ln(1+a+\zeta_n) < 0$, which is a weaker condition. We will refer to this phenomenon as the "stabilisation by noise": the solution of $x_{n+1} = x_n(1+a)$ with $a > 0$ can be stabilised by adding some noise to a (for an in-depth discussion of stabilisation by noise see e.g. [1, 3, 5, 15]). A natural question arises: when the noise is present, how fast must S_n decay to guarantee that the convergence persists? Would $S_n \to 0$ be enough? We will discuss this question at length in the present paper but the short answer is the following. The coefficients S_n must have a power law decay, with the exponent determined by the nature of the noise. Thus, the addition of the noise stabilises the homogenous linear equation but imposes stronger conditions on the free coefficient S_n of the non-homogenous one. It is interesting to compare our results with those available in the continuous case, where the interplay of the noise and the rate of decay of the free coefficient was studied in [2].

In nonlinear case (1), however, the noise does not have such stabilising effect. Our stability result (if restricted to i.i.d. noises) includes only the case $\mathbf{E}\xi_n < 0$. We investigate the case $\mathbf{E}\xi_n > 0$ further and show, for bounded i.i.d. ξ_n, that $\lim_{n\to\infty} X_n = 0$ with probability zero. Heuristically, the noise does not have the stabilising effect on the nonlinear equation because the coefficient by ξ_n becomes too small if $X_n \to 0$ (see condition (3)). The situation changes when instead of equation (1) we consider a discrete version of Ito stochastic equation with the drift and diffusion parts separated and multiplied by coefficients with different scaling:

$$X_{n+1} = (1 + kf(X_n)a + \sqrt{kf(X_n)}\zeta_{n+1})X_n + S_n, \quad n \in \mathbb{N}_0. \tag{5}$$

In this case we give a sufficient conditions for $\lim_{n\to\infty} X_n = 0$ a.s. even when a is positive (but not too large).

The structure of the paper is as follows. In section 2 we give some necessary definitions and state two lemmas. Lemma 1 can be considered a discrete version of martingale convergence theorem and is the main tool we use to prove our results. Section 3 is devoted to the a.s. convergence to zero of solutions to the linear equations with independent noises. We also present a result on the rate of decay of the solutions. Section 4 is devoted to a discussion of the obtained results as they apply to the i.i.d. noise. Further results in this simple case highlight some aspects of behaviour of the solutions. In particular, we construct some examples that indicate that our conditions for the a.s. convergence might be necessary as well as sufficient. We also find that when the decay of S_n is insufficient to guarantee convergence but $\mathbf{E}\ln(1+\xi_n)$ is negative, the lower limit of the solution is still zero. This implies that, in some cases, the solution will oscillate with increasing amplitude.

Section 5 is devoted to nonlinear equation (1). Sufficient conditions which guarantee that $\lim_{n\to\infty} X_n = 0$ are given in the case when S_n are summable and when S_n^α are summable with some $\alpha < 1$. We also prove, for bounded i.i.d. ξ_n with $\mathbf{E}\xi_n > 0$, that $\lim_{n\to\infty} X_n = 0$ with probability zero. Then we consider equation (5), a discrete version of Ito stochastic equation, and give a sufficient conditions for the a.s. convergence of the solutions to zero.

We illustrate our results with examples and defer all proofs to the last section of the paper.

2 Auxiliary Definitions and Facts

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of independent random variables. We suppose that filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated: $\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i \le n\}$. Among all sequences $\{X_n\}_{n\in\mathbb{N}}$ of random variables we distinguish those for which X_n are $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}$.

We use the standard abbreviation "*a.s.*" for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text.

A stochastic sequence $\{X_n\}_{n\in N}$ is said to be an $\mathcal{F}_n$*-martingale*, if $\mathbf{E}|X_n| < \infty$ and $\mathbf{E}\big(X_n\big|\mathcal{F}_{n-1}\big) = X_{n-1}$ a.s. for all $n \in \mathbb{N}$. A stochastic sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be an $\mathcal{F}_n$*-martingale-difference*, if $\mathbf{E}|\mu_n| < \infty$ and $\mathbf{E}\big(\mu_n\big|\mathcal{F}_{n-1}\big) = 0$ a.s. for all $n \in \mathbb{N}$.

For more details on stochastic concepts and notation we refer the reader to [14, 16, 18, 24].

Below is a version of a martingale convergence theorem, which is convenient for many proofs.

Lemma 1. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a non-negative $\mathcal{F}_n$-measurable process, $\mathbf{E}|Z_n| < \infty$ $\forall n \in \mathbb{N}$ and*

$$Z_{n+1} \le Z_n + u_n - v_n + \nu_{n+1}, \quad n \in \mathbb{N},$$

where $\{\nu_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}_n$-martingale-difference, $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ are nonnegative $\mathcal{F}_n$-measurable processes, $\mathbf{E}|u_n|$ and $\mathbf{E}|v_n|$ are finite.

Then

$$\left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} v_n < \infty\right\} \bigcap \{Z \to\}.$$

Here $\{Z \to\}$ denotes the set of all $\omega \in \Omega$ for which $Z_\infty = \lim\limits_{n\to\infty} Z_n$ exists and is finite.

We will also use the following elementary estimate.

Lemma 2. *For any $\alpha \ge 1$ there exists a function K continuous on $(0, \infty)$ such that for any $a > 0$ and $b > 0$*

$$(a + b)^\alpha \le (1 + \epsilon)a^\alpha + K(\epsilon)b^\alpha,$$

where $K(\varepsilon)$ can be estimated in the following way:

$$K(\varepsilon) \le 1 + K_1(\alpha)\varepsilon^{1-\alpha}.$$

We define $[u]^+$ and $[u]^-$ to be the positive and negative parts of u correspondingly,

$$[u]^+ = \begin{cases} u, & \text{if } u > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad [u]^- = \begin{cases} u, & \text{if } u < 0, \\ 0, & \text{otherwise.} \end{cases}$$

We will say that a sequence $\{S_n\}$ is α-summable if

$$\sum_{n=1}^{\infty} S_n^\alpha < \infty.$$

3 Linear non-homogeneous equation with independent noises.

Below is our main result on the limit of solutions to linear equation (2). The conditions for a.s. existence of a limit depend on the balance between α-summability of S_n and the signs of $\mathbf{E}(1+\xi_{i+1})^\alpha - 1$.

Theorem 1. *Let X_n be a solution to equation (2). If there exists $\alpha > 0$ such that*

$$\sum_{i=1}^{\infty} [\mathbf{E}(1+\xi_{i+1})^\alpha - 1]^+ < \infty, \tag{6}$$

and

$$\sum_{i=1}^{\infty} S_i^\alpha < \infty, \qquad \text{if} \quad \alpha \le 1, \tag{7}$$

$$\sum_{i=1}^{\infty} \frac{S_i^\alpha}{\left|1 - \mathbf{E}(1+\xi_{i+1})^\alpha\right|^{\alpha-1}} < \infty, \qquad \text{if} \quad \alpha > 1, \tag{8}$$

then $\lim_{n\to\infty} X_n$ exists. If, in addition,

$$\sum_{i=1}^{\infty} [\mathbf{E}(1+\xi_{i+1})^\alpha - 1]^- = -\infty, \tag{9}$$

then $\lim_{n\to\infty} X_n = 0$.

Remark 1. If ξ_n are independent and identically distributed (i.i.d.), as opposed to just independent, then $\mathbf{E}(1+\xi_{n+1})^\alpha - 1$ does not depend on n. Therefore conditions (6) and (9) are fulfilled whenever $\mathbf{E}(1+\xi_{n+1})^\alpha - 1 < 0$ for the corresponding value of α.

We note that if $\beta < \alpha$ then $\mathbf{E}(1+\xi_{n+1})^\alpha - 1 < 0$ implies $\mathbf{E}(1+\xi_{n+1})^\beta - 1 < 0$. Thus the requirements on ξ get stronger with the growth of α. This is compensated by weakening of the requirements on S_n (in the i.i.d. case condition (8) is just the α-summability of S_n).

Interestingly, when $\alpha < 1$ one can have $\mathbf{E}\xi_n > 0$. This will be discussed in more detail in Section 4.4 below.

The following example illustrates the case when $\sum_{i=1}^{\infty} S_i = \infty$ and $\alpha > 1$.

Example 1. Let

$$\xi_n = \begin{cases} -n^{-\frac{1}{3}} & \text{with probability } \ 1 - \frac{1}{n^2}, \\ \sqrt{n} & \text{with probability } \ \frac{1}{n^2}, \end{cases}$$

and

$$S_n \sim n^{-\frac{3}{4}}.$$

Then

$$\mathbf{E}\xi_n = n^{-\frac{1}{3}}\left(1 - \frac{1}{n^2}\right) + \sqrt{n}\frac{1}{n^2} \sim -n^{-\frac{1}{3}},$$

and

$$\mathbf{E}\xi_n^2 = -n^{-\frac{2}{3}}\left(1 - \frac{1}{n^2}\right) + n\frac{1}{n^2} \sim n^{-\frac{2}{3}}.$$

Therefore,

$$1 - \mathbf{E}(1+\xi_n)^2 = -2\mathbf{E}\xi_n - \mathbf{E}\xi_n^2 \sim 2n^{-\frac{1}{3}}.$$

Even though S_n are not summable, conditions (9) and (8) are fulfilled with $\alpha = 2$, since

$$\sum_{n=1}^{\infty}[\mathbf{E}(1+\xi_{n+1})^2 - 1] \sim -2\sum_{n=1}^{\infty} n^{-\frac{1}{3}} = -\infty,$$

$$\sum_{n=1}^{\infty} \frac{S_n^2}{1 - \mathbf{E}(1+\xi_{n+1})^2} \sim \sum_{n=1}^{\infty} n^{\frac{1}{3}} n^{-\frac{6}{4}} = \sum_{n=1}^{\infty} n^{-\frac{7}{6}} < \infty.$$

Then Theorem 1 implies that $\lim_{n\to\infty} X_n = 0$ a.s.

The next result gives the rate of decay of solutions to equation (2) when we impose more restriction on the summability of the free coefficient S_n.

Theorem 2. *Let ξ_n be independent random variables and X_n be a solution to equation (2). If for some $\alpha \in (0,1]$ there are κ_i such that*

$$\kappa_i \geq \left[\mathbf{E}(1+\xi_{i+1})^\alpha - 1\right]^-, \tag{10}$$

$$\sum_{i=1}^{\infty} \kappa_i = -\infty, \tag{11}$$

$$\sum_{n=1}^{\infty} e^{-\sum_{i=1}^{n+1}\kappa_i} S_n^\alpha < \infty,$$

then for every $\gamma \in (0,1)$

$$\lim_{n\to\infty} e^{-\gamma\sum_{i=1}^{n}\kappa_i} X_n^\alpha = 0.$$

4 Discussion of Theorem 1

In this section we limit ourselves to considering i.i.d. ξ_n. We discuss two questions here, the sharpness of the conditions of Theorem 1 and using $\mathbf{E}\ln\big(1+\xi_i\big) < 0$ as an indicator of a.s. convergence.

4.1 Is α-summability necessary?

The following lemma shows that in general one can not relax the condition of α-summability of S_n.

Lemma 3. *For any α and β satisfying $0 < \alpha < \beta$ there exist i.i.d. random variables $\{\xi_n\}_{n=1}^{\infty}$ and perturbations S_n such that*

$$\mathbf{E}(1+\xi)^{\alpha} = 1, \tag{12}$$

$$\sum_{n=1}^{\infty} S_n^{\beta} < \infty, \tag{13}$$

and yet the solution X_n of equation (2) is diverging in the sense that

$$\limsup_{n\to\infty} X_n = \infty \qquad a.s.$$

Remark 2. Theorem 1 requires $\alpha > \beta$ to guarantee a.s. convergence of X_n.

4.2 Homogeneous equation

When equation (2) is homogeneous (*i.e.* $S_n = 0$), the limit is zero if and only if $\mathbf{E}\ln\big(1+\xi_i\big) < 0$ (see, for example, [18] or [20]):

Theorem 3. *Assume that $\{\xi_n\}_{n\in\mathbb{N}}$ are i.i.d. random variables and X_n is the solution of equation (2) with $S_n = 0$. Then $\lim_{n\to+\infty} X_n = 0$ a.s. if and only if $\mathbf{E}\ln\big(1+\xi_i\big) < 0$.*

It seems, therefore, that $\mathbf{E}\ln\big(1+\xi_i\big) < 0$ is a natural indicator of the convergence of X_n. Sections 4.3 and 4.4 develop this observation.

4.3 Lower limit

When $\mathbf{E}\ln\big(1+\xi_i\big) < 0$ and S_n is non-zero but decreases exponentially with n, it was proved in [20] that $\lim_{n\to+\infty} X_n = 0$. When S_n does not decrease as rapidly, it turns out that condition $\mathbf{E}\ln\big(1+\xi_i\big) < 0$ guarantees that the lower limit of X_n is equal to zero.

Theorem 4. *Let ξ_n be i.i.d. with $\mathbf{E}\ln(1+\xi_{n+1}) < 0$. If there exists $\alpha > 0$ such that $\sum_{i=1}^{\infty} S_i^{\alpha} < \infty$, then*

$$\liminf_{n\to\infty} X_n = 0 \qquad a.s.$$

Remark 3. In some cases, in particular those covered by Lemma 3, the lower limit is equal to zero while the limit does not exist. An interesting question is the existence of the limiting distribution of X_n in such cases.

4.4 Connection between $\mathbf{E}\ln(1+\xi_i)$ and $\mathbf{E}(1+\xi_i)^\alpha - 1$.

Theorem 3 indicates that the sign of $\mathbf{E}\ln(1+\xi_i)$ is crucial in the question of stability of homogenous equation with i.i.d. noises. Theorem 1, however, depends on the sign of $\mathbf{E}(1+\xi_i)^\alpha - 1$ to establish stability. The following lemma provides the connection between the two expectations.

Lemma 4. *Let ξ be such that $\mathbb{P}(\xi > 0) > 0$. Then $\mathbf{E}\ln(1+\xi) < 0$ if and only if there exists $\alpha > 0$ such that $\mathbf{E}(1+\xi)^\alpha - 1 = 0$. If such α exists then*

$$\mathbf{E}(1+\xi)^\beta - 1 < 0 \qquad \forall \beta \in (0, \alpha). \tag{14}$$

Proof. The harder "only if" part was proved in [9], using that $\mathbf{E}(1+\xi)^\alpha$ is a convex function and its derivative at $\alpha = 0$ is equal to $\mathbf{E}\ln(1+\xi)$. Convexity also implies inequality (14).

To prove the "if" part we take expectation of the both parts of the inequality $(1+\xi)^u \geq 1 + u\ln(1+\xi)$ which can be obtained by truncating the Taylor series of $(1+\xi)^u$ with respect to u. □

When ξ_n are not identically distributed, one needs a uniform bound on α. Such a bound is provided by the following lemma.

Lemma 5. *Suppose that there exists some constant $K > 0$ such that*

$$\frac{\mathbf{E}\left[(2+\xi_i)\ln^2(1+\xi_i)\right]}{|\mathbf{E}\ln(1+\xi_i)|} \leq K, \quad \forall i \in \mathbb{N}. \tag{15}$$

Then for all α satisfying

$$\alpha < \min(1/K, 1)$$

one has

$$\alpha\mathbf{E}\ln(1+\xi_i) \leq \mathbf{E}(1+\xi_i)^\alpha - 1 \leq \alpha\left(\mathbf{E}\ln(1+\xi_i) + \frac{|\mathbf{E}\ln(1+\xi_i)|}{2}\right). \tag{16}$$

Example 2. Suppose that $-1 < -k \leq \xi_n \leq L$ and $|\mathbf{E}\ln(1+\xi_n)| \geq c$ for some $k, L, c > 0$ uniformly in $n \in \mathbb{N}$. Then condition (15) is fulfilled.

4.5 Reformulation of Theorem 1 in the i.i.d. case

Following the discussion of the previous sections we can reformulate Theorem 1 in this concise way.

Corollary 1. *Let ξ_n be i.i.d. random variables satisfying*

$$\mathbf{E}(1+\xi_n)^\alpha - 1 \leq 0$$

for some $\alpha > 0$. If S_n are α-summable then the solutions of

$$X_{n+1} = X_n\Big(1 + \xi_{n+1}\Big) + S_n, \qquad n \in \mathbb{N}_0,$$

converge to zero a.s.

Proof. The only part of the statement that does not obviously follow from Theorem 1 is what happens when $\mathbf{E}(1+\xi_n)^\alpha - 1 = 0$. In this case Theorem 1 guarantees only the existence of a limit. Here, however, we employ Lemma 4 to infer that $\mathbf{E}\ln(1+\xi_n) < 0$. Then we use Theorem 4 to confirm that the limit must indeed be zero. □

On the other hand, Lemma 3 hints that α-summability is not only a sufficient but also a necessary condition. We formulate this guess as a conjecture.

Conjecture 1. *Let ξ_n be i.i.d. random variables satisfying $\mathbf{E}\ln(1+\xi_n) < 0$ and let $\alpha > 0$ be such that*

$$\mathbf{E}(1+\xi_n)^\alpha - 1 = 0.$$

Then the solutions of

$$X_{n+1} = X_n\Big(1+\xi_{n+1}\Big) + S_n, \qquad n \in \mathbb{N}_0$$

a.s. converge to zero **if and only if** *S_n is α-summable.*

Another interesting question would be to study the convergence of X_n to zero *in probability*. For this type convergence, it might be possible to relax the conditions on the decay of S_n. Previous results by various authors [10, 4, 17] should be helpful in this direction.

5 Nonlinear equation

In this section we consider nonlinear recursion of the type

$$X_{n+1} = X_n\Big(1+f(X_n)\xi_{n+1}\Big) + S_n, \qquad n \in \mathbb{N}_0 \tag{17}$$

with independent random variables ξ_n. As mentioned earlier we assume that the function $f(u)$ is continuous with values in the interval $[0,1]$ and is equal to 0 only at $u=0$. We also assume that both terms in equation (17) are non-negative for all n.

5.1 Convergence results for nonlinear equation with independent noises

Only the $\alpha = 0$ case of Theorem 1 really carries over to the nonlinear equations of type (17).

Theorem 5. *If the components of equation (17) satisfy the conditions detailed above, S_n are summable and*

$$\sum_{n=1}^{\infty} [\mathbf{E}\xi_n]^+ < \infty, \tag{18}$$

then $\lim_{n\to\infty} X_n$ *exists. If, in addition,*

$$\sum_{n=1}^{\infty} [\mathbf{E}\xi_n]^- = -\infty, \tag{19}$$

then $\lim_{n\to\infty} X_n = 0$.

In the case when S_n are α-summable for some $\alpha \in (0, 1)$ we obtain a much more restrictive result compared with Theorem 1. Even the case of i.i.d. noises with positive $\mathbf{E}\xi_n$ is not covered by this theorem. We will explore the reason for this in the next section.

Theorem 6. *Let* S_n *be* α*-summable for some* $\alpha \in (0, 1)$,

$$\sum_{n=1}^{\infty} [\mathbf{E}\xi_n]^+ < \infty, \tag{20}$$

then $\lim_{n\to\infty} X_n$ *exists. If, in addition,* $3\mathbf{E}\xi_n^2 - (2-\alpha)[\mathbf{E}\xi_n^3]^+ > 0$ *starting with some* n *and*

$$\sum_{n=1}^{\infty} \left(\mathbf{E}\xi_n^2 - \frac{2-\alpha}{3} [\mathbf{E}\xi_n^3]^+ \right) = \infty, \tag{21}$$

then $\lim_{n\to\infty} X_n = 0$.

The following example shows that in the case when S_n^α are summable with some $\alpha < 1$, Theorem 6 gives less restrictive conditions than Theorem 5.

Example 3. Let ξ_n be uniformly distributed on the interval $[-1+n^{-2}, 1]$. Then

$$\mathbf{E}\xi_n \propto n^{-2}, \qquad \mathbf{E}\xi_n^2 \propto 1 \qquad \text{and} \qquad \mathbf{E}\xi_n^3 \propto n^{-2}.$$

Thus conditions (20) and (21) are fulfilled for all $\alpha \in (0, 1)$, but condition (19) is not.

5.2 Divergence in nonlinear equation with $\mathbf{E}\xi_i > 0$

In this subsection we present a result explaining why one cannot fully generalise Theorem 1 to nonlinear equations of the type (17).

Theorem 7. *Let* X_n *be a solution of equation (17) with i.i.d.* ξ_n *satisfying*

$$\mathbf{E}\xi_n > 0 \qquad \textit{and} \qquad -1 < -k_0 \le \xi_n \le L, \qquad n \in \mathbb{N}.$$

Then $\mathbb{P}\{X_n \to 0\} = 0$.

However, if, instead of equation (17) we consider a discrete analogue of Ito equation, the situation is reversed and we obtain a convergence result when S_n^α is summable with some $\alpha > 0$.

5.3 Analogue of Ito equation

We consider the discrete analogue of Ito equation

$$X_{n+1} = (1 + kf(X_n)a + \sqrt{kf(X_n)}\zeta_{n+1})X_n + S_n, \quad X_0 > 0, \tag{22}$$

where $a > 0$, ζ_n are i.i.d., $\mathbf{E}\zeta_n = 0$, $\mathbf{E}\zeta_n^2 < \infty$ and $\mathbf{E}|\zeta_n|^3 < \infty$.

We assume, as everywhere before, that for all positive u and all n

$$1 + kf(u)a + \sqrt{kf(u)}\zeta_{n+1} > 0 \qquad \text{and} \qquad S_n \geq 0. \tag{23}$$

Theorem 8. *Let conditions (3) and (23) be fulfilled. Let also*

$$a < \frac{\mathbf{E}\zeta^2}{2}.$$

Suppose S_n are α-summable for some α satisfying the inequality

$$\alpha < \alpha_0 = \frac{\mathbf{E}\zeta^2 - 2a}{\mathbf{E}\zeta^2}. \tag{24}$$

Then, for small enough k, $\mathbb{P}\{X_n \to 0\} = 1$.

Remark 4. We can treat equation (22) as an equation with noise $\xi_n = a + \zeta_n$ where $a = E\xi_n$. From this point of view equation (22) is a modification of equation (17), in which the coefficients of the two parts of the noise, drift and diffusion, are different. Since $a > 0$, the corresponding deterministic equation,

$$x_{n+1} = (1 + kf(x_n)a)x_n + S_n,$$

is unstable. The diffusion part, $\sqrt{kf(x_n)}\zeta_{n+1}$, stabilises the equation. It becomes possible because the coefficient of the diffusion part, $\sqrt{kf(x_n)}$, decreases slower then the coefficient of the drift part.

It is worth noting that equations (22) and (17) coincide only when $a = 0$.

6 Proofs

6.1 Deterministic lemma

For the purposes of comparison with equation (1), we discuss here a stability result for the deterministic equation

$$x_{n+1} = x_n(1 + f(x_n)a_n) + S_n, \quad x_0 > 0, \quad n \in \mathbb{N}_0.$$

Lemma 6. *Let $S_n \geq 0$, $f : \mathbb{R}^1 \to [0, 1]$, $f(0) = 0$ and $\inf_{u>c} uf(u) > 0$ $\forall c > 0$. Let also $0 > a_n > -1$ and $\sum_{n=1}^{\infty} a_n = -\infty$.*

If $\lim_{n\to\infty} S_n/a_n = 0$, then $\lim_{n\to\infty} x_n = 0$.

Proof. We note that the solution x_n remains positive for all n. We consider two possibilities: $\liminf x_n > 0$ and $\liminf x_n = 0$.

In the first case there exist $c > 0$ and N such that $x_n > c$ for all $n > N$. Let $c_1 = \inf_{u>c}\{f(u)u\}$ and $N_1 > N$ be such that $S_n \le \frac{c_1|a_n|}{2}$ for $n > N_1$. We have for $n > N_1$

$$x_{n+1} = x_{N_1} + \sum_{i=N_1}^{n} [x_i f(x_i) a_i + S_i] \le x_{N_1} + c_1 \sum_{i=N_1}^{n} \left[a_i + \frac{|a_i|}{2}\right] \le x_{N_1} - c_1 \sum_{i=N_1}^{n} \frac{|a_i|}{2}.$$

When $n \to \infty$ the right-hand-side of the inequality tends to $-\infty$, which contradicts the positivity of the solution. Thus $\liminf x_n = 0$.

Now assume that, even though $\liminf x_n = 0$, the lemma is still incorrect, i.e. $\limsup_{n\to\infty} x_n = c > 0$. We fix some $\varepsilon < c/2$ and define

$$0 < \varepsilon_1 = \inf_{\varepsilon < u < 2\varepsilon} \{f(u)u\}.$$

Now find N such that $S_n < \varepsilon_1 |a_n|/2$ and $S_n < \varepsilon$ whenever $n \ge N$.

If $x_n < \varepsilon$ (which must happen infinitely often) with $n > N$, we can estimate

$$x_{n+1} \le x_n + S_n \le 2\varepsilon.$$

If, on the other hand, $\varepsilon < x_n < 2\varepsilon$ then, by definition of ε_1, $x_n f(x_n) \ge \varepsilon_1$ and therefore

$$x_{n+1} = x_n(1 + f(x_n)a_n) + S_n \le x_n - \varepsilon_1 |a_n| + \frac{\varepsilon_1}{2}|a_n| < x_n.$$

Combining the above two facts we deduce that, once x_n gets below ε, it cannot increase past 2ε. Thus $\limsup_{n\to\infty} x_n \le 2\varepsilon < c$ and we arrive to a contradiction. □

6.2 Proof of Theorem 1

We split the proof into two parts: $\alpha \in (0, 1]$ and $\alpha > 1$.

6.2.1 Proof of Theorem 1 with $\alpha \in (0, 1]$

We note that ρ_{i+1}, defined by

$$\rho_{i+1} = X_i^\alpha (1 + \xi_{i+1})^\alpha - X_i^\alpha \mathbf{E}(1 + \xi_{i+1})^\alpha \tag{25}$$

is an $\mathcal{F}_{n+1}$-martingale-difference.

We apply Hölder inequality $(x + y)^\alpha \le x^\alpha + y^\alpha$ to equation (2) and get

$$\begin{aligned} X_{n+1}^\alpha &\le X_n^\alpha (1 + \xi_{n+1})^\alpha + S_n^\alpha \\ &= X_n^\alpha + X_n^\alpha \big(\mathbf{E}(1 + \xi_{n+1})^\alpha - 1\big) + \big[X_i^\alpha (1 + \xi_{n+1})^\alpha - X_i^\alpha \mathbf{E}(1 + \xi_{n+1})^\alpha\big] + S_n^\alpha \\ &= X_n^\alpha + X_n^\alpha \big(\mathbf{E}(1 + \xi_{n+1})^\alpha - 1\big) + S_n^\alpha + \rho_{n+1} \\ &\le X_n^\alpha + X_n^\alpha \big[\mathbf{E}(1 + \xi_{n+1})^\alpha - 1\big]^+ + S_n^\alpha + \rho_{n+1} \end{aligned}$$

with ρ_{n+1} defined in equation (25). We let

$$Y_n = e^{-\sum_{i=1}^{n}\eta_i} X_n^\alpha, \quad \text{with} \quad \eta_i = \big[\mathbf{E}(1+\xi_{i+1})^\alpha - 1\big]^+,$$

and using the above, arrive at

$$\begin{aligned} Y_{n+1} - Y_n &= e^{-\sum_{i=1}^{n+1}\eta_i}\big(X_{n+1}^\alpha - X_n^\alpha\big) + X_n^\alpha e^{-\sum_{i=1}^{n+1}\eta_i}\big(1 - e^{\eta_{n+1}}\big) \\ &\le e^{-\sum_{i=1}^{n+1}\eta_i}\big(X_n^\alpha \eta_{n+1} + \rho_{n+1} + S_n^\alpha\big) - \eta_{n+1} X_n^\alpha e^{-\sum_{i=1}^{n+1}\eta_i} \\ &= e^{-\sum_{i=1}^{n+1}\eta_i}\rho_{n+1} + e^{-\sum_{i=1}^{n+1}\eta_i} S_n^\alpha = \bar{\rho}_{n+1} + \bar{S}_n^\alpha. \end{aligned}$$

Since $\bar{\rho}_{n+1}$ is an $\mathcal{F}_{n+1}$-martingale-difference and $\sum_{i=1}^\infty \bar{S}_n^\alpha < \infty$ by a combination of conditions (6) and (7), we can apply Lemma 1. Therefore $Y_n = \exp\{-\sum_{i=1}^n \eta_i\} X_n^\alpha$ converges as $n \to \infty$. From condition (6) we infer that X_n^α also a.s. converges to a finite limit.

To prove that $\lim_{n\to\infty} X_n = 0$ we apply Lemma 1 to the inequality

$$X_{n+1}^\alpha \le X_n^\alpha + X_n^\alpha\big[\mathbf{E}(1+\xi_{n+1})^\alpha - 1\big]^- + X_n^\alpha\big[\mathbf{E}(1+\xi_{n+1})^\alpha - 1\big]^+ + S_n^\alpha + \rho_{n+1},$$

where

$$\sum_{i=0}^\infty \big[\mathbf{E}(1+\xi_{i+1})^\alpha - 1\big]^+ X_i^\alpha$$

converges a.s. due to condition (6) and the convergence of X_n^α. From Lemma 1 we infer that

$$-\sum_{i=0}^\infty \big[\mathbf{E}(1+\xi_{n+1})^\alpha - 1\big]^- X_i^\alpha$$

has to be a.s. finite. Combining it with condition (9) we conclude that $X_i^\alpha \to 0$ a.s.

6.2.2 Proof of Theorem 1 with $\alpha > 1$

Let

$$\varepsilon_n = \frac{|1 - \mathbf{E}(1+\xi_{n+1})^\alpha|}{2\mathbf{E}(1+\xi_{n+1})^\alpha}. \tag{26}$$

Applying Lemma 2 we get

$$X_{n+1}^\alpha \le (1+\varepsilon_n) X_n^\alpha \Big(1 + \xi_{n+1}\Big)^\alpha + K(\varepsilon_n) S_n^\alpha, \tag{27}$$

where $K(\varepsilon_n)$ can be estimated by the following

$$\begin{aligned} K(\varepsilon_n) &\le 1 + K(\alpha)\varepsilon_n^{1-\alpha} = 1 + K(\alpha)\frac{(2\mathbf{E}(1+\xi_{n+1})^\alpha)^{\alpha-1}}{|1-\mathbf{E}(1+\xi_{n+1})^\alpha|^{\alpha-1}} \\ &\le 1 + K(\alpha)\frac{C^{\alpha-1}}{|1-\mathbf{E}(1+\xi_{n+1})^\alpha|^{\alpha-1}} \le \frac{K_1(\alpha)}{|1-\mathbf{E}(1+\xi_{n+1})^\alpha|^{\alpha-1}}, \end{aligned}$$

where we used that $\mathbf{E}(1+\xi_{n+1})^\alpha$ is bounded due to condition (6). From equations (26)-(27) we get

$$\begin{aligned} X_{n+1}^\alpha \leq X_n^\alpha &+ X_n^\alpha\left[(1+\varepsilon_n)\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1\right] \\ &+ X_n^\alpha(1+\varepsilon_n)\left[\Big(1+\xi_{n+1}\Big)^\alpha - \mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha\right] + K(\varepsilon_n)S_n^\alpha. \end{aligned}$$

By substituting the value of ε_n into $(1+\varepsilon_n)\mathbf{E}\left(1+\xi_{n+1}\right)^\alpha - 1$ we see that it is equal to $\left[\mathbf{E}\left(1+\xi_{n+1}\right)^\alpha - 1\right]/2$ when $\mathbf{E}\left(1+\xi_{n+1}\right)^\alpha < 1$ and to $3\left[\mathbf{E}\left(1+\xi_{n+1}\right)^\alpha - 1\right]/2$ otherwise. That is, we can write

$$(1+\varepsilon_n)\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1 = \frac{1}{2}\left[\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1\right]^- + \frac{3}{2}\left[\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1\right]^+.$$

We finally arrive to

$$\begin{aligned} X_{n+1}^\alpha \leq X_n^\alpha &+ \frac{1}{2}X_n^\alpha\left[\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1\right]^- + \frac{3}{2}X_n^\alpha\left[\mathbf{E}\Big(1+\xi_{n+1}\Big)^\alpha - 1\right]^+ \\ &+ \rho_{n+1} + \frac{K_1(\alpha)}{(1-\mathbf{E}(1+\xi_{n+1})^\alpha)^{\alpha-1}}S_n^\alpha, \end{aligned}$$

where ρ_{n+1} is an $\mathcal{F}_{n+1}$-martingale-difference. Now we apply Lemma 1 and complete the proof as in Section 6.2.1.

6.3 Proof of Theorem 2

We mimic the proof of Theorem 1 (see Section 6.2.1) with $Y_n = e^{-\sum_{i=1}^n \kappa_i}X_n^\alpha$ to get

$$Y_{n+1} - Y_n \leq \bar{\rho}_{n+1} + e^{-\sum_{i=1}^{n+1}\kappa_i}S_n^\alpha.$$

Because $\bar{\rho}_{n+1}$ is an $\mathcal{F}_{n+1}$-martingale-difference and due to condition (10), we can apply Lemma 1. Hence we get that $Y_n = \exp\{-\sum_{i=1}^n \kappa_i\}X_n^\alpha$ converges to a finite limit as $n \to \infty$. Then for every $\gamma \in (0,1)$

$$\exp\left\{-\gamma\sum_{i=1}^n \kappa_i\right\}X_n^\alpha \leq Y_n \exp\left\{(1-\gamma)\sum_{i=1}^n \kappa_i\right\} \to 0 \tag{28}$$

using condition (11).

6.4 Proof of Lemma 3

We choose γ such that $\alpha < \gamma < \beta$ and take $S_n = n^{-1/\gamma}$ so that condition (13) is clearly satisfied. Now define the distribution of ξ_n so that $1+\xi_n$ takes values in (a,∞), $a > 0$, with the density function

$$p(x) = \frac{\gamma a^\gamma}{x^{1+\gamma}}.$$

First we ascertain that $\mathbf{E}(1+\xi)^\alpha = 1$. Indeed, since $\alpha < \gamma$,

$$\mathbf{E}(1+\xi)^\alpha = \gamma a^\gamma \int_a^\infty x^{-1-\gamma+\alpha} dx = \frac{\gamma}{\gamma-\alpha} a^\alpha$$

and condition (12) can be satisfied with an appropriate choice of a.

Now we can study the behaviour of solutions of equation (2). Since both summands in the right hand side of equation (2) are positive, $X_{n+1} \geq S_n$ and therefore $X_{n+2} \geq (1+\xi_{n+2})S_n$. Define the sequence of independent events $A_n = \{(1+\xi_{n+2})S_n > C\}$, where $C > 0$ is an arbitrary constant. We have

$$\mathbb{P}(A_n) = \mathbb{P}\left((1+\xi_{n+2}) > \frac{C}{S_n}\right) = \mathbb{P}\left((1+\xi_{n+2}) > Cn^{1/\gamma}\right) = \frac{a^\gamma}{C^\gamma} n^{-1}.$$

Thus,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty,$$

and, by Borel-Cantelli lemma, events A_n must happen infinitely often. Therefore, infinitely often $X_n > C$. Since C was arbitrary, we conclude that $\limsup_{n\to\infty} X_n = \infty$ a.s.

6.5 Proof of Theorem 4

Assume the contrary, for some $s > 0$ the event $J_s = \{\omega\colon \inf_n X_n^\alpha > s\}$ occurs with non-zero probability. Fix $\epsilon > 0$ such that $\alpha\mathbf{E}\ln(1+\xi) + \ln(1+\epsilon) < 0$ and consider the event $\Theta = \{\omega\colon \sum_{i=1}^n \ln\left((1+\epsilon)(1+\xi_i)^\alpha\right) \to -\infty\}$. By applying the law of large numbers it is straightforward to show that Θ occurs with probability 1.

Raising recursion (2) to power α we get by Lemma 2

$$X_{n+1}^\alpha \leq (1+\epsilon) X_n^\alpha \Big(1+\xi_{n+1}\Big)^\alpha + K(\epsilon) S_n^\alpha. \tag{29}$$

Now let n be such that $K(\epsilon)S_n^\alpha < s/2$. Restricting our attention to $\omega \in J_s$ we apply logarithm to both sides of inequality (29) and use the inequality

$$\ln(x+y) \leq \ln(x) + \frac{y}{x}$$

to obtain

$$\ln X_{n+1}^\alpha \leq \ln\Big(X_n^\alpha (1+\epsilon)(1+\xi_{n+1})^\alpha\Big) + \frac{K(\varepsilon) S_n^\alpha}{X_n^\alpha (1+\epsilon)(1+\xi_{n+1})^\alpha}.$$

Combining inequality (29) and the definition of J_s we can estimate $X_n^\alpha(1+\epsilon)(1+\xi_{n+1})^\alpha \geq X_{n+1}^\alpha - K(\epsilon)S_n^\alpha > s/2$ and, therefore,

$$\ln X_{n+1}^\alpha \leq \ln(X_n^\alpha) + \ln\Big((1+\epsilon)(1+\xi_{n+1})^\alpha\Big) + K(\epsilon)\frac{S_n^\alpha}{s/2}.$$

Applying the above inequality recursively we obtain

$$\ln X^\alpha_{n+k} < \ln(X^\alpha_n) + \sum_{i=1}^{k} \ln\Big((1+\epsilon)(1+\xi_{n+i})^\alpha\Big) + C\sum_{i=0}^{k-1} S^\alpha_{n+i},$$

where $C = 2K(\epsilon)/s$. Since $X^\alpha_{n+k} > s$ and S^α_n are summable to, say, S, we conclude that for all k

$$\sum_{i=1}^{k} \ln\Big((1+\epsilon)(1+\xi_{n+i})^\alpha\Big) > \ln(s) - \ln(X^\alpha_n) - CS$$

and, therefore, the event ω cannot belong to Θ. Thus $J_s \cap \Theta = \emptyset$ which is a contradiction.

6.6 Proof of Lemma 5

Taking the expectation of the Taylor expansion of $(1+\xi_i)^\alpha$ in terms of α we get

$$\mathbf{E}(1+\xi_i)^\alpha = 1 + \alpha\mathbf{E}\ln(1+\xi_i) + \alpha^2\mathbf{E}\left(\frac{\ln^2(1+\xi_i)}{2}(1+\xi_i)^\theta\right),$$

where $\theta \in [0,\alpha]$. The left side of estimate (16) is then obtained by leaving out the third term.

To estimate $\mathbf{E}\left(\ln^2(1+\xi_i)(1+\xi_i)^\theta/2\right)$ from above we consider two cases: $1+\xi_i > 1$ and $1+\xi_i < 1$. Since $\theta \le \alpha \le 1$, in the first case we have $(1+\xi_i)^\theta \le (1+\xi_i)$, while in the second $(1+\xi_i)^\theta \le (1+\xi_i)^0 = 1$. Then, in both cases, we have

$$(1+\xi_i)^\theta \le 2+\xi_i.$$

If $\mathbf{E}\ln(1+\xi_i)$ is negative we continue with

$$\begin{aligned}\mathbf{E}(1+\xi_i)^\alpha &\le 1 + \alpha\mathbf{E}\ln(1+\xi_i)\left(1 - \alpha\frac{\mathbf{E}\left((2+\xi_i)\ln^2(1+\xi_i)\right)}{2\,|\mathbf{E}\ln(1+\xi_i)|}\right)\\ &\le 1 + \alpha\mathbf{E}\ln(1+\xi_i)\left(1-\alpha\frac{K}{2}\right) \le 1 + \frac{\alpha}{2}\mathbf{E}\ln(1+\xi_i),\end{aligned}$$

while if $\mathbf{E}\ln(1+\xi_i) > 0$ we obtain by a similar calculation

$$\mathbf{E}(1+\xi_i)^\alpha \le 1 + \alpha\frac{3\mathbf{E}\ln(1+\xi_i)}{2}.$$

6.7 Proof of Theorem 5

We note that ρ_{i+1}, defined by

$$\rho_{i+1} = f(X_i)X_i\xi_{i+1} - f(X_i)X_i\mathbf{E}(\xi_{i+1}).$$

is an $\mathcal{F}_{n+1}$-martingale-difference.

After rearranging in equation (17) we get recursively

$$\begin{aligned}
X_{n+1} &= X_n + f(X_n)X_n\mathbf{E}\xi_{n+1} + \big[f(X_n)X_n\xi_{n+1} - f(X_n)X_n\mathbf{E}\xi_{n+1}\big] + S_n \\
&= X_n + f(X_n)X_n\big[\mathbf{E}\xi_{n+1}\big]^+ + f(X_n)X_n\big[\mathbf{E}\xi_{n+1}\big]^- + \rho_{n+1} + S_n \\
&\leq X_n + f(X_n)X_n\big[\mathbf{E}\xi_{n+1}\big]^+ + \rho_{n+1} + S_n \\
&\leq X_n + X_n\big[\mathbf{E}\xi_{n+1}\big]^+ + \rho_{n+1} + S_n.
\end{aligned} \tag{30}$$

From this point we continue as in Section 6.2.1 with $\eta_i = \big[\mathbf{E}\xi_{n+1}\big]^+$ and conclude that X_i converges to a finite limit a.s. Then

$$\sum_{i=0}^{\infty}[\mathbf{E}(\xi_{i+1})]^+ f(X_i)X_i$$

is a.s. finite. Applying Lemma 1 again (to the second line in inequality (30)), we conclude that

$$\sum_{i=0}^{\infty}[\mathbf{E}(\xi_{i+1})]^- f(X_i)X_i$$

also has to be a.s. finite. If condition (19) is fulfilled, $f(X_i)X_i$ is forced to converge to zero. Therefore $X_i \to 0$ a.s.

6.8 Proof of Theorem 6

Applying the inequality

$$(1+x)^\alpha \leq 1 + \alpha x - \frac{1-\alpha}{2}x^2 + \frac{(1-\alpha)(2-\alpha)}{6}x^3, \qquad x > -1,\ 0 < \alpha < 1 \tag{31}$$

and noting that $f^2(X_n) \geq f^3(X_n)$, we obtain

$$\begin{aligned}
&\mathbf{E}\Big[X_n^\alpha(1+f(X_n)\xi_{n+1})^\alpha \big| \mathcal{F}_n\Big] \\
&\leq X_n^\alpha\left(1 + \alpha f(X_n)\mathbf{E}\xi_{n+1} - \frac{1-\alpha}{2}f^2(X_n)\mathbf{E}\xi_{n+1}^2 + \frac{(1-\alpha)(2-\alpha)}{6}f^3(X_n)\mathbf{E}\xi_{n+1}^3\right) \\
&\leq X_n^\alpha + \alpha X_n^\alpha f(X_n)[\mathbf{E}\xi_{n+1}]^+ - \frac{1-\alpha}{2}X_n^\alpha f^2(X_n)\left(\mathbf{E}\xi_{n+1}^2 - \frac{(2-\alpha)}{3}[\mathbf{E}\xi_{n+1}^3]^+\right).
\end{aligned}$$

Now, applying inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ to equation (17) we get from the above

$$\begin{aligned}
X_{n+1}^\alpha &\le X_n^\alpha \Big(1 + f(X_n)\xi_{n+1}\Big)^\alpha + S_n^\alpha \\
&= \mathbf{E}\Big[X_n^\alpha (1+f(X_n)\xi_{n+1})^\alpha \big| \mathcal{F}_n\Big] \\
&\qquad + \Big(X_n^\alpha (1+f(X_n)\xi_{n+1})^\alpha - \mathbf{E}\Big[X_n^\alpha (1+f(X_n)\xi_{n+1})^\alpha \big|\mathcal{F}_n\Big]\Big) + S_n^\alpha \\
&\le X_n^\alpha + \alpha X_n^\alpha f(X_n)[\mathbf{E}\xi_{n+1}]^+ \\
&\qquad - \frac{1-\alpha}{2} X_n^\alpha f^2(X_n)\left(\mathbf{E}\xi_{n+1}^2 - \frac{(2-\alpha)}{3}[\mathbf{E}\xi_{n+1}^3]^+\right) + \rho_{n+1} + S_n^\alpha.
\end{aligned}$$

Now we complete the proof in the same way as in Theorem 5.

6.9 Proof of Theorem 7

For the proof we need some preliminary facts.

Lemma 7. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}_n$-measurable random variables such that $\mathbf{E}\big(X_n\big|\mathcal{F}_{n-1}\big) = 1$. Let $Z_n = \prod_{i=1}^n X_i$ and $\mathbf{E}|Z_n| < \infty$ for all $n \in \mathbb{N}$. Then $\{Z_n\}_{n\in\mathbb{N}}$ is a martingale.*

Proof. To check the martingale condition $\mathbf{E}\big(Z_n\big|\mathcal{F}_{n-1}\big) = Z_{n-1}$ we use the $\mathcal{F}_{n-1}$-measurability of Z_{n-1}:

$$\mathbf{E}\big(Z_n\big|\mathcal{F}_{n-1}\big) = \mathbf{E}\big(Z_{n-1}X_n\big|\mathcal{F}_{n-1}\big) = Z_{n-1}\mathbf{E}\big(X_n\big|\mathcal{F}_{n-1}\big) = Z_{n-1}.$$

□

Lemma 8. *Let X_n be a solution of equation (17). Then the sequence $\{M_n\}_{n\in\mathbb{N}}$, defined by*

$$M_n = \prod_{i=0}^{n-1} \frac{(1+f(X_i)\xi_{i+1})^{-1}}{\mathbf{E}\left((1+f(X_i)\xi_{i+1})^{-1}\big|\mathcal{F}_i\right)} \tag{32}$$

is an $\mathcal{F}_n$- martingale.

Proof. To make sure that our definition makes sense we estimate

$$1 + f(X_i)\xi_{i+1} \ge 1 - |\xi_{i+1}| > 1 - k_0 > 0, \tag{33}$$

therefore $\mathbf{E}\left((1+f(X_i)\xi_{i+1})^{-1}\big|\mathcal{F}_i\right)$ is well defined. Because M_n is always positive, we can write $\mathbf{E}|M_n| = \mathbf{E}M_n = \mathbf{E}M_1 = 1 < \infty$. Now we apply Lemma 7 to conclude the proof. □

The lemma below is a variant of the theorem of convergence of non-negative martingale (see e.g. [14]).

Lemma 9. *If $\{X_n\}_{n\in\mathbb{N}}$ is non-negative martingale, then $\lim_{n\to\infty} X_n$ exists with probability 1.*

From Lemma 8 and Lemma 9 we can get

Corollary 2. *Let $\{M_n\}_{n\in\mathbb{N}}$ be the martingale defined by (32), then $\lim_{n\to\infty} M_n$ exists with probability 1.*

Now we proceed to the proof of the theorem. First we note that the solution X_n of equation (17) can be represented in the following form

$$X_n = X_0 M_n^{-1} \prod_{i=0}^{n-1} \frac{1}{\mathbf{E}\left((1+f(X_i)\xi_{i+1})^{-1}\big|\mathcal{F}_i\right)}. \tag{34}$$

Here M_n is defined by equation (32) and, by Corollary 2, $M_n \le H_1$ with a.s. finite random variable $H_1 = H_1(\omega)$.

Suppose now that theorem is not correct. Then there exists a set $\Omega_1 \subseteq \Omega$ of non-zero probability such that $X_n \to 0$ a.s. on Ω_1. We aim to show that for any $\omega \in \Omega_1$, there exists $N(\omega)$ such that

$$\mathbf{E}\Big(\big(1+f(X_i)\xi_{i+1}\big)^{-1}\big|\mathcal{F}_i\Big) \le 1, \qquad \forall i \ge N(\omega).$$

For $\forall i \in \mathbb{N}$ we can perform the Taylor expansion

$$(1+f(X_i)\xi_{i+1})^{-1} = 1 - f(X_i)\xi_{i+1} + f^2(X_i)\xi_{i+1}^2 - \frac{f^3(X_i)\xi_{i+1}^3}{(1+\theta_{i+1})^4}$$

with θ_{i+1} lying between 0 and $f(X_i)\xi_{i+1}$. Using equation (33) and noting that X_n is positive and

$$\mathbf{E}\big(f(X_i)\big|\mathcal{F}_i\big) = f(X_i), \quad \mathbf{E}\big(\xi_{i+1}\big|\mathcal{F}_i\big) = \mathbf{E}\xi_{i+1}, \quad 0 \le f(X_i) \le 1,$$

we estimate

$$\mathbf{E}\left(\frac{f^3(X_i)\xi_{i+1}^3}{(1+\theta_{i+1})^4}\bigg|\mathcal{F}_i\right) \le \frac{L^3 f^3(X_i)}{(1-k_0)^4}.$$

Then we have

$$\begin{aligned}\mathbf{E}\left(\big(1+f(X_i)\xi_{i+1}\big)^{-\alpha}\bigg|\mathcal{F}_i\right) &\le 1 - f(X_i)\mathbf{E}\xi_{i+1} + f^2(X_i)L^2 + \frac{L^3 f^3(X_i)}{(1-k_0)^4} \\ &= 1 - f(X_i)\left(\mathbf{E}\xi_{i+1} - f(X_i)L^2 - \frac{L^3 f^2(X_i)}{(1-k_0)^4}\right).\end{aligned}$$

The function f is such that $f(X_n) \to 0$ a.s. on Ω_1, therefore we can find such $N(\omega)$ that for Ω_1 and $i \ge N(\omega)$

$$\mathbf{E}\left((1+f(X_i)\xi_{i+1})^{-\alpha}\bigg|\mathcal{F}_i\right) \le 1 - f(X_i)\frac{\mathbf{E}\xi_{i+1}}{2} < 1.$$

Combining this with representation (34) and with a.s. boundedness of M_n we conclude that solution X_n cannot tend to 0 on Ω_1.

6.10 Proof of Theorem 8

As before, we raise equation (22) to power α and set

$$\rho_{n+1} = X_n^\alpha \left(1 + kf(X_n)a + \sqrt{kf(X_n)}\zeta_{n+1}\right)^\alpha - \mathbf{E}\left[X_n^\alpha \left(1 + kf(X_n)a + \sqrt{kf(X_n)}\zeta_{n+1}\right)^\alpha \middle| \mathcal{F}_n\right].$$

We now aim to show that the conditional expectation above is negative. Applying inequality (31) and remembering that $\mathbf{E}\zeta_{n+1} = 0$, we get

$$\begin{aligned}
&\mathbf{E}\left[\left(1 + kf(X_n)a + \sqrt{kf(X_n)}\zeta_{n+1}\right)^\alpha \middle| \mathcal{F}_n\right] \\
&\quad \le 1 + \alpha kf(X_n)a - \frac{\alpha(1-\alpha)}{2}\left((akf(X_n))^2 + kf(X_n)\mathbf{E}\zeta^2\right) \\
&\qquad + \frac{\alpha(1-\alpha)(2-\alpha)}{6}\left((akf(X_n))^3 + 3a(kf(X_n))^2\mathbf{E}\zeta^2 + (kf(X_n))^{3/2}\mathbf{E}\zeta^3\right) \\
&\quad \le 1 + \alpha kf(X_n)\left(a - \frac{1-\alpha}{2}\mathbf{E}\zeta^2 + O(\sqrt{k})\right)
\end{aligned}$$

Due to condition (24) there exist k_0 and a_0, such that for $k < k_0$

$$a - \frac{1-\alpha}{2}\mathbf{E}\zeta^2 + O(\sqrt{k}) \le -a_0 < 0.$$

Therefore we obtain the estimation

$$X_{n+1}^\alpha \le X_n^\alpha - a_0 X_n^\alpha \alpha kf(X_n) + \rho_{n+1} + S_n^\alpha.$$

Now we can apply Lemma 1 and complete the proof by the familiar method.

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