Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Determine a unit vector that has the same direction as the vector $3\vec{\imath} - 4\vec{\jmath}$.

2. Determine the vector projection of the vector $\langle 0, 1\rangle$ onto the vector $\langle 2, 3\rangle$.

3. Find either a vector equation or a parametric equation for the line that passes through the point with coordinates $(1, 7)$ and is parallel to the vector $2\vec{\imath} + 6\vec{\jmath}$.

4. Sketch the graph of a function f having the following properties: $\lim_{x\to-\infty} f(x) = 2$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to 0^+} f(x) = 1$.

5. Given the information that $f(1) = 2$, $f'(1) = 3$, $g(1) = 4$, and $g'(1) = 5$, determine the value of the derivative $(fg)'(1)$.

6. If a TI-89 calculator is tossed upward on the moon with an initial velocity of 10 meters per second, then its height in meters after t seconds is equal to $10t - 0.83\,t^2$. Determine the velocity of the calculator after 1 second.

7. (a) State the precise definition of what $\lim_{x\to a} f(x) = L$ means (using ϵ and δ).

   (b) Use the precise definition of limit to prove that $\lim_{x\to 2} 5x = 10$.

8. The function $2/x$ is never equal to 0 (because a fraction can equal 0 only if the numerator equals 0). This function, however, takes the value -1 when $x = -2$ and the value $+1$ when $x = +2$. Since 0 is a value in between -1 and $+1$, why doesn't this contradict the Intermediate Value Theorem?

9. (a) Write the definition of the derivative $f'(x)$ in terms of a limit.

   (b) Use the limit definition of the derivative (not the power rule) to show that
   $$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

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10. **Optional problem for extra credit**

Kim, Lee, and Datta are asked to determine the value of the limit

$$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+1}}.$$

Kim argues as follows: "If x has large magnitude, then x^2+1 is nearly equal to x^2, and $\sqrt{x^2} = x$, so the fraction is nearly equal to $2x/x$, and the limit must be 2."

Lee argues as follows: "I remember the trick is to divide the numerator and the denominator both by the highest power of x, which is x^2; then we will have $2x/x^2$ in the numerator, which has limit 0, so the answer must be 0."

Datta argues as follows: "I tried evaluating the function on my calculator. When $x = -10$, I got a function value of -1.99007; when $x = -100$, I got $y = -1.9999$; and when $x = -1000$, I got $y = -2$. When I tried negative values of x of even larger magnitude, I kept getting $y = -2$. Therefore the limit must be -2."

Decide who (if anyone) is correct, and explain the shortcomings in the arguments proposed by these three students.