Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Recall from section 3.11 that $P(x) = f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2$ is the quadratic approximation to the function f at the point a. Use l'Hospital's rule to show that if f has a continuous second derivative, then
$$\lim_{x\to a} \frac{f(x) - P(x)}{(x-a)^2} = 0.$$

   Solution. Since $f(a) - P(a) = 0$, l'Hospital's rule implies that
$$\lim_{x\to a} \frac{f(x) - P(x)}{(x-a)^2} = \lim_{x\to a} \frac{f'(x) - P'(x)}{2(x-a)} = \lim_{x\to a} \frac{f'(x) - f'(a) - f''(a)(x-a)}{2(x-a)}.$$

   The new limit is again an indeterminate 0/0 form, and a second application of l'Hospital's rule gives the result
$$\lim_{x\to a} \frac{f''(x) - f''(a)}{2} = 0.$$

2. The TI-89 calculator says that $\tan^{-1}(\tan(\pi)) = 0$. Since $\tan^{-1}$ and tan are inverse functions, why is the answer not equal to π?

   Solution. Since the tangent function is periodic, it is not one-to-one, and it does not have an inverse function on its whole domain. The function $\tan^{-1}$ is the inverse of the *restriction* of the tangent function to the interval $(-\pi/2, \pi/2)$.

   In other words, $\tan^{-1}(x)$ means the angle *between* $-\pi/2$ *and* $\pi/2$ whose tangent is equal to x. Thus $\tan^{-1}(\tan(\pi))$ means the angle between $-\pi/2$ and $\pi/2$ whose tangent equals $\tan(\pi)$, and since the tangent function has period π, this angle is 0.

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3. Show that $\sin^2(\cos^{-1}(x)) = 1 - x^2$ when $-1 \leq x \leq 1$.
   [Remember that the two exponents have different meanings: the exponent -1 means inverse function, while the exponent 2 means the second power.]

   Solution. Method 1. Draw a right triangle having angle θ, hypotenuse equal to 1, and side adjacent to angle θ equal to x. Then $\theta = \cos^{-1}(x)$. The third side of the triangle equals $\sqrt{1-x^2}$ by the Pythagorean theorem, but this third side also equals $\sin(\theta)$. The quantity to be found is $\sin^2(\theta)$, which therefore equals $1 - x^2$.

   Method 2. Since $\sin^2(\theta) = 1 - \cos^2(\theta)$ for every θ, we can write

   $$\sin^2(\cos^{-1}(x)) = 1 - \cos^2(\cos^{-1}(x)).$$

   Now $\cos^2(\cos^{-1}(x))$ means $[\cos(\cos^{-1}(x))]^2$, which simplifies to x^2 because cos and $\cos^{-1}$ are inverse functions. Substituting back into the displayed formula gives the desired result $1 - x^2$.

4. The TI-89 calculator says that $\lim_{x\to\infty}(xe^{1/x} - x) = 1$. Prove this result.

   Solution. Method 1. Write $xe^{1/x} - x = x(e^{1/x} - 1) = \dfrac{e^{1/x} - 1}{1/x}$. When $x \to \infty$, we have an indeterminate $0/0$ form, so l'Hospital's rule equates the limit to

   $$\lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

   Method 2. Write $x = 1/t$. Then

   $$\lim_{x\to\infty} x(e^{1/x} - 1) = \lim_{t\to 0^+} \frac{e^t - 1}{t},$$

   and by l'Hospital's rule, the new limit equals $\lim_{t\to 0^+} e^t = 1$.