Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

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Highlights

From last time:

- The determinant tells whether a matrix is invertible.
- One way to compute a determinant: use row operations.
- Another way to compute a determinant: use cofactor expansion.

Today:

- Vector spaces and subspaces: definitions and examples.

Definition and first examples of vector spaces

A *vector space* is a set of mathematical objects equipped with two operations: addition and multiplication by scalars (for us, usually the real numbers) satisfying the usual commutative, associative, and distributive laws. There should be an additive identity element **0**, and each element should have an additive inverse, and the scalar 1 should be a multiplicative identity.

The formal axioms are listed in the textbook on page 119.

Basic examples.

- the Euclidean plane R^2
- Euclidean 3-space R^3
- Euclidean n-space R^n
- the space of $m \times n$ matrices $R^{m \times n}$

More examples: function spaces

- The set P of all polynomials is a vector space. In this space, "vectors" are objects like $x^8 + \frac{5}{2}x^4 - 7x + \sqrt{3}$.
- Fix a counting number n. The set P_n of all polynomials with degree less than n is a vector space. This example is a *subspace* of the preceding space.
- Fix a closed interval $[a, b]$. The set $C[a, b]$ of all continuous functions on the interval is a vector space. In this space, "vectors" are objects like $5e^x + |x| \cos(x)$.
- The set $C^2[a, b]$ of all functions with a continuous second derivative is a vector space. This example is a subspace of the preceding space.

Examples that fail to be vector spaces

- The set of polynomials of degree 2 is not a vector space. Not closed under addition: $(1+x^2)+(x-x^2)=(1+x)$; the degree does not stay equal to 2. Also, the set lacks an additive identity element, because the polynomial 0 does not have degree 2.
- The set of polynomials with integral coefficients is not a vector space. The set is not closed under scalar multiplication; for instance, multiplication by $\sqrt{2}$ does not preserve the set.
- The set of solutions of the differential equation $y''+5y'+4y=\sin(x)$ is not a vector space. Not closed under addition: the sum of two solutions satisfies the equation $y''+5y'+4y=2\sin(x)$.

In general, a sub**set** of a vector space is a sub**space** if and only if it is closed under both addition and scalar multiplication.

Nullspace of a matrix

If A is an $m\times n$ matrix, then the *nullspace* $N(A)$ is the set of vectors $\mathbf{x}$ in R^n such that $A\mathbf{x}=\mathbf{0}$.
The nullspace is always a subspace of R^n.

Example (#4b, p. 132). $A=\begin{pmatrix}1 & 2 & -3 & -1\\ -2 & -4 & 6 & 3\end{pmatrix}$. Find $N(A)$.

To solve $A\mathbf{x}=\mathbf{0}$, row reduce the augmented matrix:

$$\left(\begin{array}{cccc|c}1 & 2 & -3 & -1 & 0\\ -2 & -4 & 6 & 3 & 0\end{array}\right)\xrightarrow{R_2+2R_1}\left(\begin{array}{cccc|c}1 & 2 & -3 & -1 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right)$$

So $x_4=0$, x_2 and x_3 are free variables, and $x_1=-2x_2+3x_3$.
The null space consists of all linear combinations of the vectors $\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}3\\0\\1\\0\end{pmatrix}$, the so-called *span* of those vectors.

Spanning sets

Example: exercise 11, page 133

Which of the vectors $\begin{pmatrix}2\\6\\6\end{pmatrix}$ and $\begin{pmatrix}-9\\-2\\5\end{pmatrix}$ is in the span of the vectors $\begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\4\\2\end{pmatrix}$? Restatement:

Is the system $x_1\begin{pmatrix}-1\\2\\3\end{pmatrix}+x_2\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}2\\6\\6\end{pmatrix}$ consistent?

Is the system $x_1\begin{pmatrix}-1\\2\\3\end{pmatrix}+x_2\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}-9\\-2\\5\end{pmatrix}$ consistent?

Use row operations to find out.

Exercise 11 continued

$$\left(\begin{array}{cc|c}-1 & 3 & 2\\ 2 & 4 & 6\\ 3 & 2 & 6\end{array}\right)\xrightarrow[R_3+3R_1]{R_2+2R_1}\left(\begin{array}{cc|c}-1 & 3 & 2\\ 0 & 10 & 10\\ 0 & 11 & 12\end{array}\right)\xrightarrow{R_2/10}\left(\begin{array}{cc|c}-1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 11 & 12\end{array}\right)$$

$$\xrightarrow{R_3-11R_2}\left(\begin{array}{cc|c}-1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right)\quad\text{inconsistent.}$$

$$\left(\begin{array}{cc|c}-1 & 3 & -9\\ 2 & 4 & -2\\ 3 & 2 & 5\end{array}\right)\xrightarrow[R_3+3R_1]{R_2+2R_1}\left(\begin{array}{cc|c}-1 & 3 & -9\\ 0 & 10 & -20\\ 0 & 11 & -22\end{array}\right)\xrightarrow{R_2/10}\left(\begin{array}{cc|c}-1 & 3 & -9\\ 0 & 1 & -2\\ 0 & 11 & -22\end{array}\right)$$

$$\xrightarrow{R_3-11R_2}\left(\begin{array}{cc|c}-1 & 3 & -9\\ 0 & 1 & -2\\ 0 & 0 & 0\end{array}\right)\quad\text{consistent.}$$