Math 304

Linear Algebra

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Highlights

Reminder. The first examination is Friday, June 9.

From last time:

- vector spaces and subspaces
- the nullspace of a matrix
- spanning sets

Today:

- linear independence

Linear dependence

Example. The span of the vectors $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\1\end{pmatrix}$ is all of the space R^2. In fact, the span of any two of these three vectors is R^2. The three vectors are *linearly dependent*: in fact, they satisfy the relation

$$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

In general, a set of vectors is called linearly dependent if

- one of the vectors is in the span of the others,
- or, equivalently, if a non-trivial linear combination of the vectors equals the **0** vector.

Geometric interpretation

Two vectors in R^3 are linearly dependent if they lie in the same line.
Three vectors in R^3 are linearly dependent if they lie in the same plane.

Example. The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in R^3 are *linearly independent* because they do not lie in a plane. The span of the vectors is all of R^3.

Equivalent conditions for linear independence

The follow conditions are equivalent:

- The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.
- The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$.
- The matrix A that has the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as columns has trivial nullspace: $N(A) = \{\mathbf{0}\}$.

If the matrix A is square (the n vectors lie in the space R^n), then other equivalent conditions are: A is invertible; $\det(A) \neq 0$.

Example in a space of functions

Problem. Show that $\sin(x)$, $\cos(x)$, and e^x are linearly independent functions in the space $C^2[0,1]$.

Solution. Suppose there were constants c_1, c_2, c_3 such that

$$c_1 \sin(x) + c_2 \cos(x) + c_3 e^x = 0.$$

Taking the first and second derivatives shows that also

$$\begin{aligned} c_1 \cos(x) - c_2 \sin(x) + c_3 e^x &= 0 \\ -c_1 \sin(x) - c_2 \cos(x) + c_3 e^x &= 0. \end{aligned}$$

Think of these three equations as a system of equations for the unknown vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

Example continued

The vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is in the nullspace of the matrix

$$\begin{pmatrix} \sin(x) & \cos(x) & e^x \\ \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \end{pmatrix}.$$

Compute the *Wronskian determinant* by row reducing:

$$\begin{vmatrix} \sin(x) & \cos(x) & e^x \\ \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \end{vmatrix} \overset{R_3+R_1}{=} \begin{vmatrix} \sin(x) & \cos(x) & e^x \\ \cos(x) & -\sin(x) & e^x \\ 0 & 0 & 2e^x \end{vmatrix}$$

$$= 2e^x \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = 2e^x(-\sin^2 x - \cos^2 x) = -2e^x \neq 0.$$

Therefore the nullspace of the matrix is trivial, so the functions $\sin(x)$, $\cos(x)$, and e^x are linearly independent.