Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

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Highlights

From Wednesday:

- basis and dimension
- transition matrix for change of basis

Today:

- row space and column space
- the rank-nullity property

Example

Let $A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 7 & 1 \\ 4 & 8 & 5 & 11 & 3 \end{pmatrix}$.

Three-part problem. Find a basis for

- the *row space* (the subspace of R^5 spanned by the rows of the matrix)
- the *column space* (the subspace of R^3 spanned by the columns of the matrix)
- the nullspace (the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$)

We can analyze all three parts by Gaussian elimination, *even though row operations change the column space.*

Example continued

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 7 & 1 \\ 4 & 8 & 5 & 11 & 3 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that in each step, the second column is twice the first column; also, the second column is the sum of the third and fifth columns.

Row operations *change* the column space but *preserve* linear relations among the columns. The final row-echelon form shows that the column space has dimension equal to 2, and the first and third columns are linearly independent.

Example: interpretation

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 7 & 1 \\ 4 & 8 & 5 & 11 & 3 \end{pmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The dimension of the row space (called the *rank*) is 2.
- The dimension of the column space also equals the rank. Both equal the number of lead 1's in the row echelon form.
- One basis for the row space is the pair of vectors $(1, 2, 0, -1, 2)^T$ and $(0, 0, 1, 3, -1)^T$.
- One basis for the column space is the first and third columns of the *original* matrix: namely, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

Example: nullspace

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 7 & 1 \\ 4 & 8 & 5 & 11 & 3 \end{pmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The lead variables are x_1 and x_3. The other variables x_2, x_4, and x_5 are free variables (they can take arbitrary values). The nullspace consists of vectors of the form

$$\begin{pmatrix} -2x_2 + x_4 - 2x_5 \\ x_2 \\ -3x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The nullspace has dimension 3, and one basis is the vectors $(-2, 1, 0, 0, 0)^T$, $(1, 0, -3, 1, 0)^T$, and $(-2, 0, 1, 0, 1)^T$.

Rank and nullity

The example illustrates the following general principles.

- rank = dimension of row space
  = dimension of column space
  = number of lead 1's in row echelon form
- *nullity* = dimension of nullspace
  = number of free variables in row echelon form
- rank + nullity = number of columns in matrix