Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

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Highlights

From last time:

- row space and column space
- the rank-nullity property

Today:

- examples of linear transformations

What is a linear transformation?

Matrix multiplication preserves linear combinations, namely,

$$A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}), \text{ and}$$
$$A(c\mathbf{x}) = cA(\mathbf{x}) \qquad \text{for every scalar } c.$$

Any mapping/function/transformation/operator from one vector space to another is called *linear* if it satisfies the same two properties.

Examples of linear transformations

- Geometric examples in R^2 or R^3: dilation, rotation, reflection, projection
- Examples in spaces of functions: differentiation, integration, point evaluation
- Examples in the space of matrices: taking the transpose, taking the *trace* (the sum of the numbers on the main diagonal)

Non-examples of linear transformations

- Geometric: translation
- Function spaces: squaring a function
- Matrices: taking the inverse

Subspaces associated to a linear transformation

Suppose $L : V \to W$ is a linear transformation from one vector space V to another vector space W.

- The *kernel* of L is the set of vectors $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{0}$.
  Example: If $L : P_5 \to R$ is defined by $L(p) = p(0)$, then the kernel of L is the set of polynomials with constant term 0.
  The kernel is essentially the same concept as the nullspace of a matrix.
  The kernel is always a subspace of V.
  L is *one-to-one* (or *injective*) if $\ker(L) = \{\mathbf{0}\}$.
- The *range* $L(V)$ is the set of all vectors $\mathbf{w}$ in W such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v}$ in V.
  Example: If $L : P_5 \to P_5$ is differentiation, then the range of L is P_4.
  The range is always a subspace of W. The range is analogous to the column space of a matrix.
  L is *onto* (or *surjective*) if $L(V) = W$.