Math 304

Linear Algebra

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Highlights

From last time:

- scalar product and orthogonality

Today:

- orthogonal subspaces

Reinterpretation of matrix multiplication

The entries of a matrix product are the *scalar products* of the rows of the first matrix with the columns of the second matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

In particular, the nullspace of a matrix is the set of all vectors orthogonal to the row space.

Notation. If S is a subspace of R^n, then the set of all vectors orthogonal to every vector in S is the *orthogonal complement* of S, written $S^\perp$ (pronounced "S perp").

Rank-nullity revisited

An $m \times n$ matrix A determines a linear transformation from R^n into R^m.
Notation. $N(A)$ denotes the nullspace, and $R(A)$ denotes the range.

$$\begin{aligned} \text{column space} &= R(A) \\ \text{row space} &= R(A^T) \\ \text{nullspace } N(A) &= R(A^T)^\perp \\ N(A^T) &= R(A)^\perp \end{aligned}$$

The rank-nullity theorem restated: $\dim N(A) + \dim R(A^T) = n$.

Orthogonal subspaces in general

If S is any subspace of R^n, and $S^\perp$ is the orthogonal subspace, then

- $(S^\perp)^\perp = S$
- $\dim S + \dim S^\perp = n$
- $R^n = S \oplus S^\perp$
  The notation $\oplus$, called *direct sum*, means that every vector in R^n can be written in a *unique* way as a sum of an element of S and an element of $S^\perp$.

Application. If A is an $m \times n$ matrix, then

$$R^n = N(A) \oplus R(A^T)$$

and A maps the row space $R(A^T)$ one-to-one onto the column space $R(A)$.