# Math 304

## Linear Algebra

Harold P. Boas

boas@tamu.edu

June 28, 2006

## Highlights

From last time:

- application of eigenvectors to systems of differential equations

Today:

- diagonalization of matrices and applications

## When is a matrix diagonal?

Suppose a linear operator $L$ on $R^3$ is represented in a basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ by the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

This means that $L\mathbf{u}_1 = 2\mathbf{u}_1$ and $L\mathbf{u}_2 = 3\mathbf{u}_2$ and $L\mathbf{u}_3 = 5\mathbf{u}_3$. In other words, the basis vectors are eigenvectors of the operator $L$.

A square matrix $A$ is *diagonalizable* if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ can be represented in some basis by a diagonal matrix; in other words, if there is a basis consisting of eigenvectors of $A$; equivalently, if there is an invertible matrix $S$ such that $S^{-1}AS$ is a diagonal matrix.

## Example: exercise 1(b), p. 340

Diagonalize the matrix $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$. In other words, find an invertible matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$ or $A = SDS^{-1}$.

**Solution.** First find the eigenvalues and eigenvectors of $A$. The vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is an eigenvector with eigenvalue 1, and the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue 2. The matrix $S = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$ is the transition matrix from the eigenvector basis to the standard basis, and the matrix $S^{-1}AS$ is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

## Continuation

If $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$, find $A^{10}$.

**Solution.** Since $S^{-1}AS = D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and $D^{10} = \begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix}$, we have

$A^{10} = SD^{10}S^{-1} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} =$
$\begin{pmatrix} -3 + 2^{12} & -6 + 3 \times 2^{11} \\ 2 - 2^{11} & 4 - 3 \times 2^{10} \end{pmatrix}$.

**More.** Since the exponential function is given by a power series ($e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$), we also have
$e^A := I + A + \frac{1}{2!}A^2 + \cdots = Se^DS^{-1} =$
$\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} =$
$\begin{pmatrix} -3e + 4e^2 & -6e + 6e^2 \\ 2e - 2e^2 & 4e - 3e^2 \end{pmatrix}$.

## Application to differential equations

We have two ways to solve the system of differential equations
$\mathbf{y}' = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \mathbf{y}$.

From yesterday, we can write the general solution as
$\mathbf{y}(t) = c_1 e^t \begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Alternatively, we can write the general solution as
$\mathbf{y}(t) = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Se^{tD}S^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$
$\begin{pmatrix} -3e^t + 4e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & 4e^t - 3e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

In the second form, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$.