Linear Algebra

Write your **name**: Answer Key (2 points).

In **problems 1–5**, circle the correct answer. (5 points each)

1. On the vector space of polynomials, differentiation is a linear operator.

   `True` `False`

   Solution. True: the derivative of a sum is the sum of the derivatives, and the derivative of a scalar times a polynomial is the scalar times the derivative of the polynomial.

2. If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, then the vector $\mathbf{b}$ must be in the space $N(A)^{\perp}$. `True` `False`

   Solution. False. If A is an $m \times n$ matrix, then $\mathbf{b}$ is in R^m and $N(A)^{\perp}$ is a subspace of R^n, so the statement does not even make sense when $m \neq n$. What is true is that $\mathbf{b}$ must be in the range $R(A)$, and that space is equal to the space $N(A^T)^{\perp}$ (not $N(A)^{\perp}$).

3. The matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is the matrix representation (with respect to the standard basis) of the linear operator that reflects each vector $\mathbf{x}$ in R^2 about the x_2 axis and then rotates it 90° in the counterclockwise direction. `True` `False`

   Solution. True: the image of the first basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, the first column of the matrix, and the image of the second basis vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, the second column of the matrix.

4. The two functions $\sqrt{3}\,x$ and $\sqrt{5}\,(4x^2 - 3x)$ are an orthonormal set in the space $C[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

   `True` `False`

   Solution. True. The first function is normalized since $\int_0^1 (\sqrt{3}\,x)^2\,dx = \int_0^1 3x^2\,dx = [x^3]_0^1 = 1$. The second function is normalized because $\int_0^1 (\sqrt{5}\,(4x^2 - 3x))^2\,dx = \int_0^1 5(16x^4 - 24x^3 + 9x^2)\,dx = 5(\frac{16}{5} - 6 + 3) = 1$. The two functions are orthogonal because $\int_0^1 (\sqrt{3}\,x)(\sqrt{5}\,(4x^2 - 3x))\,dx = \sqrt{15}\int_0^1 (4x^3 - 3x^2)\,dx = 0$.

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5. Every invertible matrix is diagonalizable. `True` `False`

   Solution. False. For example, the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but cannot be diagonalized because the only eigenvalue is 1, and the corresponding eigenspace is spanned by the single eigenvector $(1,0)^T$. The matrix does not admit a basis of eigenvectors.

In **problems 6–9**, fill in the blanks. (7 points per problem)

6. The matrix $\begin{pmatrix} \boxed{\frac{1}{\sqrt{2}}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \boxed{\frac{1}{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is an orthogonal matrix.

7. The angle between the vectors $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ \boxed{-4} \end{pmatrix}$ in R^3 is 45°.

8. The eigenvalues of the matrix $\begin{pmatrix} 2 & 4 \\ 3 & \boxed{6} \end{pmatrix}$ are 0 and $\boxed{8}$.

9. If a 7×11 matrix A has a nullspace of dimension 5, then the nullspace of the transpose matrix A^T has dimension $\boxed{1}$.

In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points each)

10. Suppose $\mathbf{u}_1 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$. If $\mathbf{x} = 4\mathbf{u}_1 + 3\mathbf{u}_2$, find numbers c_1 and c_2 such that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

    Solution. This is a problem about change of basis, but it can be solved from first principles. The problem amounts to solving the system

$$\begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

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You can multiply out the right-hand side and solve by row reduction, or alternatively multiply by an inverse matrix (this amounts to the change of basis formula) to get

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -15 \\ 11 \end{pmatrix}.$$

11. Find a least-squares solution of the system $\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.

Solution. The associated least-squares problem is

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \qquad \text{or}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}.$$

Therefore $x_1 = 7/3$ and $x_2 = 10/14 = 5/7$.

12. The matrices $\begin{pmatrix} 2 & a & -9 \\ -4 & 2 & -6 \\ -2 & -5 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ are similar. Find the value of the number a.

Solution. There are several ways to see that $a = 11$. One way is to observe that similar matrices have equal determinants: set the determinant of the first matrix equal to 0 and solve for a. Alternatively, observe that similar matrices have the same eigenvalues: in this case, 3, 0, and 4. So row reduce the matrix $A - 3I$ and find what condition on a guarantees a non-trivial nullspace; or row reduce $A - 0I$ or $A - 4I$. Another method is to observe that the first matrix must have rank 2 (since the second matrix has rank 2), so one of the columns must be a linear combination of the other columns. One can see by inspection that the middle column of the first matrix must be the first column minus the third column.