Linear Algebra

1. Solve the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 7 \\ 3x_2 - 2x_3 &= -8 \\ -2x_1 + x_3 &= -2 \end{aligned}$$

for the variables x_1, x_2, and x_3.

Solution. Gaussian elimination (add 2 times the first row to the third row, then divide the second row by 3, then add 4 times the second row to the third row, then multiply the third row by 3) leads to the equivalent system

$$\begin{aligned} x_1 - 2x_2 + \phantom{\tfrac{2}{3}}x_3 &= 7 \\ x_2 - \tfrac{2}{3}x_3 &= -\tfrac{8}{3} \\ x_3 &= 4. \end{aligned}$$

Back substitution shows that $(x_1, x_2, x_3) = (3, 0, 4)$.

2. For most values of the parameter a, the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 7 \\ 3x_2 - 2x_3 &= -8 \\ -2x_1 + ax_3 &= -2 \end{aligned}$$

has one and only one solution for the variables x_1, x_2, and x_3. What is the one exceptional value of a for which something different happens?

Solution. Gaussian elimination (the same steps as in the preceding problem) leads to the equivalent system

$$\begin{aligned} x_1 - 2x_2 + \phantom{\tfrac{2}{3}}x_3 &= 7 \\ x_2 - \tfrac{2}{3}x_3 &= -\tfrac{8}{3} \\ (a - \tfrac{2}{3})x_3 &= \tfrac{4}{3}. \end{aligned}$$

When $a = \frac{2}{3}$, the third equation becomes the impossible equation $0 = \frac{4}{3}$. Thus the system is *inconsistent* when $a = \frac{2}{3}$, which is the exceptional value of the parameter a.