Linear Algebra

1. Let L be the linear operator on the space P_3 of polynomials of degree less than 3 such that $L(p(x)) = p''(x) + xp'(x)$. Find the matrix that represents L with respect to the ordered basis $[1, x, 1 + x^2]$.

   Solution. One computes that $L(1) = 0$, $L(x) = x$, and $L(1 + x^2) = 2 + 2x^2 = 2(1 + x^2)$. Thus the operator L maps each basis element into a multiple of itself. The representing matrix is therefore the diagonal matrix
   $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}$. Suppose A represents a linear operator L on R^2 with respect to the standard basis, and B represents the operator L with respect to a basis $[\mathbf{u}_1, \mathbf{u}_2]$. If $\mathbf{u}_1 = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} a \\ 1 \end{pmatrix}$, what is the value of a?

   Solution. Method 1 (computationally intensive): The transition matrix U from the $\mathbf{u}$-basis to the standard basis equals $\begin{pmatrix} 7 & a \\ 2 & 1 \end{pmatrix}$, and $B = U^{-1}AU$. Working out this matrix product gives an equation (actually four equivalent equations) for the parameter a, and one can solve to get $a = 3$.

   Method 2 (more conceptual): Observe that $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$, and what this equation in $\mathbf{u}$-coordinates means is that $L(\mathbf{u}_1) = -3\mathbf{u}_1 + 8\mathbf{u}_2$. On the other hand, one has in standard coordinates that $L(\mathbf{u}_1) = A\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Therefore $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 7 \\ 2 \end{pmatrix} + 8\begin{pmatrix} a \\ 1 \end{pmatrix}$. Hence $8\begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \end{pmatrix}$, so $a = 3$.