Linear Algebra

1. Suppose L is the linear operator on R^2 defined by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$. Find the matrix that represents this transformation with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$, where $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{u_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. [1(a), p. 204]

   Solution. Since $L(\mathbf{u_1}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \mathbf{u}_2$, and $L(\mathbf{u_2}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{u}_1$, the required matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   An alternate solution, which is correct but unnecessarily long, is to observe that $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents L with respect to the standard basis, $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the transition matrix from the **u**-basis to the standard basis, $U^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$ is the transition matrix from the standard basis to the **u**-basis, and the required matrix is $U^{-1}AU$. Multiplying out this matrix product indeed gives the same answer $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2. The first four *Legendre polynomials* are $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = \frac{1}{2}(3x^2 - 1)$, and $p_3(x) = \frac{1}{2}(5x^3 - 3x)$. The ordered set $[p_0, p_1, p_2, p_3]$ forms a basis for the space of polynomials of degree 3 or less. Find the matrix that represents—with respect to this basis—the operator of differentiation.

   Solution. Observe that $p_0'(x) = 0$, $p_1'(x) = 1 = p_0(x)$, $p_2'(x) = 3x = 3p_1(x)$, and $p_3'(x) = \frac{1}{2}(15x^2 - 3) = 5p_2(x) + p_0(x)$. Therefore the matrix that represents the operator of differentiation is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$