

Linear Algebra

Both problems concern the inner-product space $C[0, 1]$ of continuous functions on the interval $[0, 1]$ with the following inner product:

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx.$$

1. For which positive number a does the function $f(x) = ax^4$ have norm equal to 1?

Solution. Since $\|f\|^2 = \langle f, f \rangle = \int_0^1 a^2 x^8 dx = \left[\frac{a^2 x^9}{9} \right]_0^1 = a^2/9$, we want $a^2/9 = 1$, so $a = 3$.

2. Find the cosine of the angle between the functions $f(x) = \sqrt{x}$ and $g(x) = 1$.

Solution. Since $\langle f, g \rangle = \|f\| \|g\| \cos(\theta)$, compute:

$$\begin{aligned}\langle f, g \rangle &= \int_0^1 \sqrt{x} dx = \left[\frac{x^{3/2}}{3/2} \right]_0^1 = \frac{2}{3}, \\ \|f\|^2 &= \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}, \quad \text{so } \|f\| = \frac{1}{\sqrt{2}}, \\ \|g\|^2 &= \int_0^1 1^2 dx = 1, \quad \text{so } \|g\| = 1.\end{aligned}$$

Therefore

$$\cos(\theta) = \frac{\langle f, g \rangle}{\|f\| \|g\|} = \frac{2/3}{1/\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

[The question did not ask for the value of θ , which is $\arccos(2\sqrt{2}/3)$. Your calculator will give you an approximate value of either 0.34 radians or 19.5 degrees.]