Linear Algebra

1. Use the Gram-Schmidt process to orthonormalize the pair of functions 1 and x in the space $C[0,1]$, where the inner product is given by $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

   Solution. Since $\|1\|^2 = \int_0^1 1^2\,dx = 1$, the first function is already normalized. Now $\langle 1, x\rangle = \int_0^1 x\,dx = \frac{1}{2}$, so the Gram-Schmidt process says to replace the function x by the function $x - \frac{1}{2}$ to get a function orthogonal to the function 1. It remains to normalize this new function. Since $\|x - \frac{1}{2}\|^2 = \int_0^1 (x - \frac{1}{2})^2\,dx = \int_0^1 (x^2 - x + \frac{1}{4})\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$, the norm $\|x - \frac{1}{2}\| = \frac{1}{2\sqrt{3}}$. Therefore the final orthonormal pair will be 1 and $2\sqrt{3}(x - \frac{1}{2})$, or equivalently 1 and $\sqrt{3}(2x - 1)$.

2. If $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$, find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. [#2(a), p. 281]

   Solution. Divide the first column by $\sqrt{2}$ to normalize it. The scalar product of the second column with the new first column is $-\frac{3}{\sqrt{2}} + \frac{5}{\sqrt{2}} = \sqrt{2}$. Subtract $\sqrt{2}$ times the first column from the second column to get $\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \sqrt{2}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Normalize the new second column by dividing it by $4\sqrt{2}$. The resulting matrix is the orthogonal matrix $Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Tracking the operations that were performed shows that $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{pmatrix}$. Thus $\begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{pmatrix}$.