Linear Algebra

1. The vector $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & a & -1 \end{pmatrix}$. Find the value of a.

   Solution. Since $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & a & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -a-2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, the eigenvalue λ must be equal to 1 (look at the second component of each vector). Therefore $-a-2=2$ (look at the third component of each vector), so $a=-4$.

2. The matrix $\begin{pmatrix} 3 & 2 \\ 1 & b \end{pmatrix}$ has the number 2 as one of its eigenvalues. Determine the value of b.

   Solution. Since 2 is an eigenvalue, $0 = \det \begin{pmatrix} 3-2 & 2 \\ 1 & b-2 \end{pmatrix} = b-4$, so $b=4$.

   An equivalent but less direct approach is to find the characteristic polynomial: $\lambda^2 - (3+b)\lambda + 3b - 2 = 0$. Substitute $\lambda = 2$, and solve for b.

   An alternative solution is that the matrix $\begin{pmatrix} 3-2 & 2 \\ 1 & b-2 \end{pmatrix}$ must have non-trivial nullspace. Row reduce:

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & b-2 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & b-4 & 0 \end{array}\right)$$

   From the reduced matrix, one sees that there is a non-trivial nullspace if and only if $b=4$.