Linear Algebra

1. Suppose $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Solution. The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$ implements the operation of subtracting 4 times row 1 from row 3, so $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.
We can take $U = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$.

2. Suppose $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & -1 \\ 2 & 1 & a \end{pmatrix}$. There is exactly one value of the parameter a for which the matrix A is singular (that is, fails to be invertible). Determine that special value of the parameter a.

Solution. One method is to use elementary row operations and see what condition on a results in a row of zeroes at the bottom: namely,

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & 2 & -1 \\ 2 & 1 & a \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 6 & -5 \\ 0 & 5 & a-4 \end{pmatrix} \xrightarrow{R_3-\frac{5}{6}R_2} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 6 & -5 \\ 0 & 0 & a+\frac{1}{6} \end{pmatrix}.$$

When $a = -\frac{1}{6}$, there is a row of zeroes, so that is the special value of a.
Another method is to try to compute an inverse matrix by applying Gauss-Jordan reduction to the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & a & 0 & 0 & 1 \end{array}\right).$$

You will find that at some point you need to divide by the quantity $6a+1$, and this again shows that the special case occurs when $a = -\frac{1}{6}$.
As we will learn tomorrow, yet another method is to find what value of a makes the determinant of the matrix A equal to 0.