Linear Algebra

1. Let S be the subspace of R^3 spanned by the vectors $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

   Find $S^\perp$, the orthogonal complement of S.

   [This is exercise 3(b) on page 233.]

   Solution. We seek vectors $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ such that

   $$\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

   We can find such vectors by Gaussian elimination:

   $$\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 1 & -1 & 2 & 0\end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & -3 & 1 & 0\end{array}\right).$$

   From the echelon form, we see that x_3 is a free variable, $x_2 = x_3/3$, and $x_1 = -5x_3/3$. Consequently, $S^\perp$ consists of all vectors of the form $x_3\begin{pmatrix}-5/3\\1/3\\1\end{pmatrix}$, where x_3 is arbitrary. In other words, $S^\perp$ is the span of the vector $\begin{pmatrix}-5/3\\1/3\\1\end{pmatrix}$.

   Multiplying the basis vector by a nonzero scalar gives an equivalent answer, so you could also say that $S^\perp$ is the span of the vector $\begin{pmatrix}-5\\1\\3\end{pmatrix}$, which is a simpler answer.

   Notice that S is a two-dimensional subspace of R^3, and $S^\perp$ is a one-dimensional subspace of R^3: the dimensions of the orthogonal subspaces add up to the dimension of the whole space.

Linear Algebra

2. Find a least squares solution of the linear system

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$

Solution. Our method for solving least squares problems is to multiply by the transposed matrix $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix}$. The result is the new system

$$\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}.$$

One way to finish the solution is to multiply by the inverse matrix $\frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ to obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 11 \\ -3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 19 \\ -4 \end{pmatrix}.$$

Thus $x_1 = 19/9$ and $x_2 = -4/9$.

An alternative way to complete the solution is to use Gaussian elimination:

$$\left(\begin{array}{cc|c} 5 & -1 & 11 \\ -1 & 2 & -3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} -1 & 2 & -3 \\ 5 & -1 & 11 \end{array}\right) \xrightarrow{R_2 \to R_2 + 5R_1} \left(\begin{array}{cc|c} -1 & 2 & -3 \\ 0 & 9 & -4 \end{array}\right).$$

From the second row, we see that $x_2 = -4/9$, and back substitution shows that $x_1 = 19/9$, as before.