Linear Algebra

1. Let S be the subspace of R^3 spanned by the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find two vectors that are orthogonal to each other and that form a basis for $S^\perp$. [This is a variation on exercise 2 on page 233.]

Solution. There are infinitely many possible answers, one of which is the pair of vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Here is one way to find that answer.

We seek vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ orthogonal to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, in other words, vectors such that $x_1 - x_2 + x_3 = 0$. There are two free variables, and we can get one suitable vector by taking $x_3 = 0$ and $x_2 = 1$, in which case $x_1 = 1$. Now we have the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ as one of the basis vectors for $S^\perp$.

It remains to find a vector that is orthogonal to both of the vectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. If you remember the vector cross product from vector calculus, you could find the required vector that way. Using the methods of our course, however, you would find the required third vector by computing the null space of the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, since that null space consists of vectors orthogonal to both rows of the matrix. The augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$ has the reduced echelon form $\left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right)$, from which you can read off that the required third vector is $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Linear Algebra

2. In the space $C[0, 1]$ of continuous functions with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, find the vector projection of e^x onto x.

Solution. The required vector projection equals

$$\left(\frac{\langle e^x, x\rangle}{\|x\|^2}\right) x.$$

Now $\|x\|^2 = \langle x, x\rangle = \int_0^1 x^2\,dx = [x^3/3]_0^1 = 1/3$, and integration by parts shows that

$$\langle e^x, x\rangle = \int_0^1 xe^x\,dx = \left[xe^x - \int e^x\,dx\right]_0^1 = [xe^x - e^x]_0^1 = 1.$$

Therefore the vector projection of e^x on x is equal to $3x$.