Linear Algebra

1. Suppose
$$A = \begin{pmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \begin{pmatrix} 1/3 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Determine the value of a.

   Solution. Multiply the two indicated matrices:
$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3a+4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
The matrix on the right-hand side is supposed to be the identity matrix, so $3a + 4 = 0$, or $a = -4/3$. (Alternatively, one could determine the value of a by using the algorithm for computing an inverse matrix.)

2. Write the column vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

   Solution. We want to solve the equation
$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
for x_1 and x_2. An equivalent problem is to solve the linear system
$$\begin{aligned} x_1 + 3x_2 &= -3 \\ 2x_1 + 4x_2 &= 2. \end{aligned}$$
Subtracting 2 times the first equation from the second equation gives the equivalent system
$$\begin{aligned} x_1 + 3x_2 &= -3 \\ -2x_2 &= 8. \end{aligned}$$
Consequently, $x_2 = -4$, and back substitution shows that $x_1 = 9$. Thus we can write
$$9 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 4 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$