Math 304

Linear Algebra

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Highlights

From last time:

- application of eigenvectors to systems of differential equations

Today:

- diagonalization of matrices and applications

A visit to Diagon Alley

Suppose a linear operator L on R^3 is represented in a basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ by the diagonal matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.
This means that $L\mathbf{u}_1 = 2\mathbf{u}_1$ and $L\mathbf{u}_2 = 3\mathbf{u}_2$ and $L\mathbf{u}_3 = 5\mathbf{u}_3$.
In other words, the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are eigenvectors of the operator L.

A square matrix A is *diagonalizable* if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ can be represented in some basis by a diagonal matrix; in other words, if there is a basis consisting of eigenvectors of A; equivalently, if there is an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

Example

Diagonalize the matrix $A = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$. In other words, find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$ or, equivalently, $A = SDS^{-1}$.

Solution. First find the eigenvalues and eigenvectors of A.
Yesterday we saw that $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue -2.
The matrix $S = \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix}$ is the transition matrix from the eigenvector basis to the standard basis, and the matrix $S^{-1}AS$ is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.

Continuation

If $A = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$, find the power A^{1000}.

Solution. Since $S^{-1}AS = D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, and $D^{1000} = \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix}$, we have $A^{1000} = SD^{1000}S^{-1} =$

$$\begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix} \left(-\frac{1}{3}\right) \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} =$$

$$\left(-\frac{1}{3}\right) \begin{pmatrix} -4 + 2^{1000} & -4 + 4 \times 2^{1000} \\ 1 - 2^{1000} & 1 - 4 \times 2^{1000} \end{pmatrix}.$$

More. Since the exponential function is given by a power series ($e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$), we can write

$$e^A := I + A + \frac{1}{2!}A^2 + \cdots = Se^DS^{-1} = S \begin{pmatrix} e^1 & 0 \\ 0 & e^{-2} \end{pmatrix} S^{-1}$$

$$= \left(-\frac{1}{3}\right) \begin{pmatrix} -4e + e^{-2} & -4e + 4e^{-2} \\ e - e^{-2} & e - 4e^{-2} \end{pmatrix}.$$

Application to differential equations

We have two ways to solve the system of differential equations $\mathbf{y}' = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \mathbf{y}$.

(a) From yesterday, we can write the general solution as $\mathbf{y}(t) = c_1 e^t \begin{pmatrix} 4 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) With a different choice of c_1 and c_2, we can write $\mathbf{y}(t) = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Se^{tD}S^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$

$$\left(-\frac{1}{3}\right) \begin{pmatrix} -4e^t + e^{-2t} & -4e^t + 4e^{-2t} \\ e^t - e^{-2t} & e^t - 4e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In the second form, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$.