Linear Algebra

Write your **name**: ______________________________ (2 points).

In **problems 1–5**, circle the correct answer. (5 points per problem)

1. There exists a 6×4 matrix of rank 5. `True` `False`

   Solution. False. The rank equals both the dimension of the row space and the dimension of the column space. Since our matrix has 4 columns, the dimension of the column space is at most 4, so the rank of the matrix is at most 4.

2. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ is in the orthogonal complement of the null space of A^T. `True` `False`

   Solution. True. The orthogonal complement of the null space of A^T equals the range of A, that is, the column space of A. The linear system is consistent if and only if the vector $\mathbf{b}$ is in the column space of A.

3. In R^3, the projection of a vector $\mathbf{v}$ onto a vector $\mathbf{w}$ has length less than or equal to the length of $\mathbf{v}$. `True` `False`

   Solution. True. This statement is a general property of projections in any inner product space (not just R^3). The length of the projection of $\mathbf{v}$ onto $\mathbf{w}$ is $|\langle \mathbf{v}, \frac{\mathbf{w}}{\|\mathbf{w}\|} \rangle|$, and by the Cauchy-Schwarz inequality this quantity does not exceed $\|\mathbf{v}\|$.

4. The formula $L(p) = \int_0^1 p(x)\sin(x)\,dx$ defines a linear transformation from the space P_3 of polynomials of degree less than 3 into the one-dimensional vector space R. `True` `False`

   Solution. True. What needs to be checked is that $L(p_1 + p_2) = L(p_1) + L(p_2)$ for all polynomials p_1 and p_2, and $L(cp) = cL(p)$ for every polynomial p and every scalar c. Both properties are true because integration is a linear operation.

5. In the space $C[-1, 1]$ of continuous functions on the interval $[-1, 1]$ equipped with the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

the functions x and x^2 are orthogonal. `True` `False`

Solution. True. What needs to be checked is that $\int_{-1}^{1} x^3\,dx = 0$, and you can either evaluate the integral explicitly or observe that it equals 0 for symmetry reasons.

In **problems 6–9**, fill in the blanks. Some of the problems may have non-unique answers. (7 points per problem)

6. The angle between the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \square \\ -1 \\ 0 \end{pmatrix}$ in R^3 is equal to $\dfrac{\pi}{3}$ radians (or 60 degrees).

   Solution. Let the missing entry be a. The scalar product of the two vectors equals $1 \cdot a + 0 \cdot (-1) + 1 \cdot 0$, which simplifies to a. The scalar product also equals the product of the lengths of the vectors times the cosine of the angle between them, or $\sqrt{2} \cdot \sqrt{a^2 + 1 + 0} \cdot \frac{1}{2}$. Therefore
   $$a = \frac{\sqrt{2}}{2}\sqrt{a^2+1}.$$
   In particular, a is a *positive* number. Squaring both sides shows that $a^2 = \frac{1}{2}(a^2+1)$, or $a^2 = 1$, so $a = \pm 1$. But since a has to be positive, the value -1 is spurious, and the correct answer is $a = 1$.

7. If $L\colon R^2 \to R^2$ is a linear operator such that $L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $L\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, then the transformation L is represented (with respect to the standard basis) by the matrix $\begin{pmatrix} \square & -1 \\ \square & \square \end{pmatrix}$.

   Solution. The linearity tells us that
   $$L\begin{pmatrix} 2 \\ 0 \end{pmatrix} = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} + L\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix},$$

so $L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$. This vector is the first column of the matrix.

Similarly,

$$L\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\left\{L\begin{pmatrix}1\\1\end{pmatrix} - L\begin{pmatrix}1\\-1\end{pmatrix}\right\} = \frac{1}{2}\left\{\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}5\\6\end{pmatrix}\right\} = \begin{pmatrix}-1\\-1\end{pmatrix}.$$

This vector is the second column of the matrix.

Thus the required matrix is $\begin{pmatrix}4 & -1\\5 & -1\end{pmatrix}$.

8. The matrix $\begin{pmatrix}1/2 & \square\\ \square & 1/2\end{pmatrix}$ is an orthogonal matrix.

Solution. To make the columns have length 1, we should take the missing entries to be $\pm\sqrt{3}/2$. To make the columns orthogonal to each other, we should take the missing entries to have opposite signs. Thus there are two correct answers:

$$\begin{pmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}.$$

9. The inconsistent linear system $\begin{pmatrix}1 & 1\\1 & 0\\0 & \square\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}6\\3\\0\end{pmatrix}$ has the least-squares solution $\begin{pmatrix}4\\1\end{pmatrix}$.

Solution. Call the missing entry a. To obtain the associated least-squares problem, we multiply by the transpose matrix to get

$$\begin{pmatrix}2 & 1\\1 & 1+a^2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}9\\6\end{pmatrix}.$$

Since $\begin{pmatrix}4\\1\end{pmatrix}$ is a solution, we have

$$\begin{pmatrix}2 & 1\\1 & 1+a^2\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}9\\6\end{pmatrix}, \qquad \text{or} \qquad \begin{cases} 8+1=9,\\ 4+1+a^2=6.\end{cases}$$

The second equation tells us that $a^2 = 1$, so $a = \pm 1$. There are two correct answers: either $a = 1$ or $a = -1$.

In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points per problem)

10. Find an orthonormal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Solution. By performing Gaussian elimination, you can determine that this matrix has rank 2. Hence the column space is 2-dimensional. By inspection, you can see that each pair of columns is a linearly independent set and hence forms a basis for the 2-dimensional column space. Pick your favorite two columns and run the Gram-Schmidt algorithm to orthonormalize them.

If you pick the first two columns, you should get the answer

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Less work is required if you pick the second two columns, for they are already orthogonal to each other. You need only normalize them to get the answer

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Other answers are possible.

11. Suppose $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find a matrix S such that $S^{-1}AS = B$.

Solution. The matrix S is the transition matrix from the eigenvector basis to the standard basis. The columns of S are simply the eigenvectors. The given information tells us that the eigenvalues are 1, 2, and 3, so it remains to compute the corresponding eigenvectors.

For eigenvalue 1, we need to find the null space of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

By inspection, the vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ spans the null space.

For eigenvalue 2, we need to find the null space of the matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By inspection, the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ spans the null space.

For eigenvalue 3, we need to find the null space of the matrix

$$\begin{pmatrix} -2 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

By inspection, the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ spans the null space.

Thus

$$S = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Other answers are possible. You could multiply each column by any nonzero scalar (possibly a different scalar for each column).

Linear Algebra

12. Find the general solution of the following linear system of first-order differential equations:

$$\begin{aligned} y_1' &= y_1 + 4y_2 \\ y_2' &= -3y_1 - 6y_2 \end{aligned}$$

Solution. In matrix form, the system says $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ -3 & -6 \end{pmatrix} \mathbf{y}$. We need to find the eigenvalues and the corresponding eigenvectors of the matrix. The characteristic equation for the eigenvalues is $\lambda^2 + 5\lambda + 6 = 0$, or $(\lambda + 2)(\lambda + 3) = 0$, so the eigenvalues are -2 and -3.

For eigenvalue -2, we find the corresponding eigenvector by looking for the null space of the matrix

$$\begin{pmatrix} 3 & 4 \\ -3 & -4 \end{pmatrix},$$

and evidently the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ will do.

For the eigenvalue -3, we find the corresponding eigenvector by looking for the null space of the matrix

$$\begin{pmatrix} 4 & 4 \\ -3 & -3 \end{pmatrix},$$

and evidently the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will do.

Therefore the general solution for $\mathbf{y}$ has the form

$$c_1 e^{-2t} \begin{pmatrix} 4 \\ -3 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants. Written out in component form, the solution is

$$\begin{aligned} y_1 &= 4c_1 e^{-2t} + c_2 e^{-3t}, \\ y_2 &= -3c_1 e^{-2t} - c_2 e^{-3t}. \end{aligned}$$

Another way to solve this problem is to use the exponential matrix.