Linear Algebra

Write your **name**: ______________________________ (2 points).

In **problems 1–5**, circle the correct answer. (5 points per problem)

1. There exists a 6×4 matrix of rank 5. `True` `False`

2. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ is in the orthogonal complement of the null space of A^T. `True` `False`

3. In R^3, the projection of a vector $\mathbf{v}$ onto a vector $\mathbf{w}$ has length less than or equal to the length of $\mathbf{v}$. `True` `False`

4. The formula $L(p) = \int_0^1 p(x)\sin(x)\,dx$ defines a linear transformation from the space P_3 of polynomials of degree less than 3 into the one-dimensional vector space R. `True` `False`

5. In the space $C[-1,1]$ of continuous functions on the interval $[-1,1]$ equipped with the inner product
$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$
the functions x and x^2 are orthogonal. `True` `False`

In **problems 6–9**, fill in the blanks. Some of the problems may have non-unique answers. (7 points per problem)

6. The angle between the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \square \\ -1 \\ 0 \end{pmatrix}$ in R^3 is equal to $\dfrac{\pi}{3}$ radians (or 60 degrees).

7. If $L\colon R^2 \to R^2$ is a linear operator such that $L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $L\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, then the transformation L is represented (with respect to the standard basis) by the matrix $\begin{pmatrix} \square & -1 \\ \square & \square \end{pmatrix}$.

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8. The matrix $\begin{pmatrix} 1/2 & \boxed{} \\ \boxed{} & 1/2 \end{pmatrix}$ is an orthogonal matrix.

9. The inconsistent linear system $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & \boxed{} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$ has the least-squares solution $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

You may use the space below for scratch work.

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In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points per problem)

10. Find an orthonormal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

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11. Suppose $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find a matrix S such that $S^{-1}AS = B$.

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12. Find the general solution of the following linear system of first-order differential equations:

$$\begin{aligned} y_1' &= y_1 + 4y_2 \\ y_2' &= -3y_1 - 6y_2 \end{aligned}$$