Math 323

Linear Algebra

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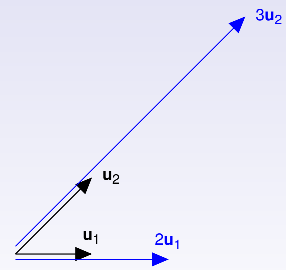
Example of eigenvectors

The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ defines a linear transformation of R^2 that is easy to understand. The transformation stretches the vector $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of 2 and stretches the vector $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by a factor of 3.



The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are called *eigenvectors*, and the scale factors 2 and 3 are the corresponding *eigenvalues*.

The transformation is particularly simple to describe in the basis $[\mathbf{u}_1, \mathbf{u}_2]$: namely, the matrix $U^{-1}AU$ is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Computing eigenvectors

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has 3 as one of its eigenvalues. Find a corresponding eigenvector.

Solution. We seek a vector $\mathbf{v}$ such that $A\mathbf{v} = 3\mathbf{v}$. Equivalently, $\mathbf{v}$ should be in the nullspace of the matrix $A - 3I$ (I = identity). Find the nullspace by row reduction:

$$\left(\begin{array}{rrr|r} 9 & 4 & -5 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 10R_1]{R_2 \to R_2 + 8R_1} \left(\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right) \xrightarrow[\text{steps}]{\text{three}} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then $\mathbf{v} = (1, -1, 1)^T$ is an eigenvector with eigenvalue 3.

Computing eigenvalues (continued)

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has other eigenvalues besides the number 3. Find them.

Solution. The condition for a number λ to be an eigenvalue of A is that the matrix $A - \lambda I$ has a non-trivial nullspace. Equivalently, $\det(A - \lambda I) = 0$, the *characteristic equation*:

$$0 = \begin{vmatrix} 12-\lambda & 4 & -5 \\ -8 & 0-\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix} \overset{R_1 \to}{\underset{R_1+R_2}{=}} \begin{vmatrix} 4-\lambda & 4-\lambda & 0 \\ -8 & -\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix}$$

$$\overset{C_2 \to}{\underset{C_2 - C_1}{=}} \begin{vmatrix} 4-\lambda & 0 & 0 \\ -8 & 8-\lambda & 5 \\ 10 & -6 & -3-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 8-\lambda & 5 \\ -6 & -3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2 - 5\lambda + 6) = (4-\lambda)(\lambda-3)(\lambda-2).$$

Therefore the eigenvalues of A are 4, 3, and 2.

Exercise

If $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$, find eigenvectors corresponding to the eigenvalues 2 and 4.

We already know that $(1, -1, 1)^T$ is an eigenvector with eigenvalue 3.

Answer. Vector $(4, -3, 4)^T$ is an eigenvector with eigenvalue 4, and $(5, -5, 6)^T$ is an eigenvector with eigenvalue 2.

Remark. $S = \begin{pmatrix} 4 & 1 & 5 \\ -3 & -1 & -5 \\ 4 & 1 & 6 \end{pmatrix}$ is the transition matrix from an eigenvector basis to the standard basis, and $S^{-1}AS$ is a diagonal matrix with the eigenvalues 4, 3, and 2 on the diagonal.