Complex Variables

Instructions Solve any **seven** of the following eight problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Explain why $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z}\,dz = 0.$

   Solution. One reason is that the function $z^{-1}\sin(z)$ has a removable singularity, since $\lim_{z\to 0} z^{-1}\sin(z) = 1$, so the integral equals 0 by Cauchy's theorem.

   Alternatively, Cauchy's integral formula implies that the integral equals $2\pi i \sin(0)$, which reduces to 0.

   You could also apply the Residue Theorem, observing that the integrand has residue equal to 0 at the origin.

2. State **two** of the following four theorems:

   - Morera's theorem
   - Liouville's theorem
   - Rouché's theorem
   - Schwarz's lemma.

   Solution. The statements are in the textbook on pages 129, 130, 177, and 193.

3. Give an example of a function that is analytic in the punctured plane (meaning the set $\{z : z \neq 0\}$) and that has a simple pole when $z = 0$, a double zero when $z = 1$, and no other zeroes or poles.

   Solution. The simplest example is the rational function $\dfrac{(z-1)^2}{z}$.

4. The function $\dfrac{1}{\sin(z)}$ has a Laurent series expansion in powers of z and z^{-1} valid when $0 < |z| < \pi$. Determine the first two nonzero terms of this expansion.

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Solution. Since $\sin(z) = z - \frac{1}{6}z^3 + O(z^5)$, it follows that

$$\frac{1}{\sin(z)} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{6}z^2 + O(z^4)} = \frac{1}{z} \cdot \left(1 + \frac{1}{6}z^2 + O(z^4)\right)$$
$$= \frac{1}{z} + \frac{1}{6}z + O(z^3).$$

I used the binomial series trick: $\frac{1}{1-u} = 1 + u + u^2 + \cdots$ when $|u| < 1$. You could also do explicit long division.

5. The function $\dfrac{\cos(z)}{z^3}$ has a pole of order 3 when $z = 0$. Determine the residue of this function at the pole.

Solution. Since $\cos(z) = 1 - \frac{1}{2}z^2 + O(z^4)$, it follows that

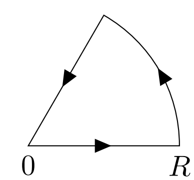
$$\frac{\cos(z)}{z^3} = \frac{1}{z^3} - \frac{1/2}{z} + O(z),$$

so the residue (the coefficient of the $1/z$ term in the Laurent series) equals $-1/2$.

Alternatively, you could use the formula for the residue at a multiple pole to compute the residue as follows:

$$\frac{1}{2!} \cdot \frac{d^2}{dz^2}\left[z^3 \cdot \frac{\cos(z)}{z^3}\right]_{z=0} = \frac{1}{2} \cdot \frac{d^2}{dz^2}\cos(z)\,\bigg|_{z=0} = \frac{1}{2}(-\cos(0)) = -\frac{1}{2}.$$

6. The TI-89 calculator says that $\displaystyle\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{\pi}{3}$. Prove this formula. Suggestion: integrate over a "piece of pie" of angle $\pi/3$.



Solution. If γ is the illustrated contour, then there is one pole inside (at $e^{\pi i/6}$), so

$$\int_\gamma \frac{1}{1+z^6}\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^6}; e^{\pi i/6}\right) = \frac{2\pi i}{6(e^{\pi i/6})^5} = \frac{\pi i}{3}e^{-5\pi i/6}.$$

On the other hand, we can parametrize the three parts of the contour respectively by $z = x$ (where x goes from 0 to R), $z = Re^{i\theta}$ (where

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θ goes from 0 to $\pi/3$), and $z = te^{\pi i/3}$ (where t goes from R to 0). Therefore the contour integral equals

$$\int_0^R \frac{1}{1+x^6}\,dx + \int_0^{\pi/3} \frac{1}{1+R^6e^{6i\theta}} Rie^{i\theta}\,d\theta + \int_R^0 \frac{1}{1+t^6} e^{\pi i/3}\,dt.$$

The middle integral is $O(1/R^5)$ because

$$\left|\frac{1}{1+R^6e^{6i\theta}} Rie^{i\theta}\right| \le \frac{R}{R^6-1} \qquad \text{(when } R > 1\text{)}.$$

Putting the pieces together, we find that

$$\frac{\pi i}{3} e^{-5\pi i/6} = \left(1 - e^{\pi i/3}\right) \int_0^R \frac{1}{1+x^6}\,dx + O(1/R^5).$$

Taking the limit as $R \to \infty$ shows that

$$\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{\pi i}{3} \cdot \frac{e^{-5\pi i/6}}{1-e^{\pi i/3}}.$$

Now $ie^{-5\pi i/6} = i(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) = -\frac{\sqrt{3}}{2}i + \frac{1}{2}$, and $1 - e^{\pi i/3} = 1 - (\frac{1}{2} + \frac{\sqrt{3}}{2}i) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, so the answer indeed reduces to $\frac{\pi}{3}$.

Alternatively, you could rewrite the problem as $\displaystyle\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^6}$ and use a semi-circular contour. Then you have to compute three residues (at $e^{\pi i/6}$, $e^{3\pi i/6}$, and $e^{5\pi i/6}$). Passing to the limit, you get the answer

$$\frac{1}{2} \cdot 2\pi i \left(\frac{1}{6e^{5\pi i/6}} + \frac{1}{6e^{15\pi i/6}} + \frac{1}{6e^{25\pi i/6}} \right),$$

which again simplifies to $\frac{\pi}{3}$.

7. The Fundamental Theorem of Algebra implies that the polynomial $3z^{28} - 2z^8 + 7z^5 + 1$ has 28 zeroes in the complex plane (counting multiplicities). How many of these 28 zeroes lie in the unit disc (the set where $|z| < 1$)? Explain how you know.

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Solution. The idea is to apply Rouché's theorem. Suppose $f(z) = -7z^5$ and $g(z) = 3z^{28} - 2z^8 + 7z^5 + 1$. On the unit circle where $|z| = 1$, we have

$$|f(z) + g(z)| = |3z^{28} - 2z^8 + 1| \leq 3 + 2 + 1 = 6 < 7 = |f(z)|.$$

Thus the hypothesis of Rouché's theorem is satisfied on the boundary circle, and we deduce that the functions $f(z)$ and $g(z)$ have the same number of zeroes inside the circle. Since $f(z)$ has a zero of order 5 at the origin, it follows that our original polynomial $g(z)$ has 5 zeroes in the unit disc (counting multiplicity).

8. Student Max conjectures that if f and g are entire functions such that $|f(z)| \leq |g(z)|$ when $|z| = 1$, then $|f(z)| \leq |g(z)|$ when $|z| \leq 1$. If Max's conjecture is correct, then prove it; otherwise, supply a counterexample showing that Max is wrong.

   Solution. Max's conjecture is wrong. Indeed, if $f(z)$ is the constant function 1 and $g(z) = z$, then $|f(z)| = |g(z)|$ when $|z| = 1$, but $|f(z)| > |g(z)|$ for every point z such that $|z| < 1$.

   Nonetheless, Max's conjecture can be salvaged by adding a supplementary hypothesis. If the function $g(z)$ has no zeroes in the closed unit disc, then Max's statement does hold. Indeed, in this case the quotient $f(z)/g(z)$ is analytic, and its modulus is at most 1 on the boundary circle, so the maximum-modulus principle implies that its modulus is at most 1 everywhere in the unit disc.