Complex Variables

Instructions Solve any **eight** of the following ten problems. Explain your reasoning in complete sentences to maximize credit.

1. The TI-89 calculator says, reasonably enough, that

$$\lim_{x\to 0} \left((x-1)^{1/3} - 1\right)^3 = -8.$$

Somewhat surprisingly, Maple and Mathematica say instead that

$$\lim_{x\to 0} \left((x-1)^{1/3} - 1\right)^3 = 1.$$

Use complex numbers to explain how the two different answers both can be justified mathematically.

Remark. The Maple command is `limit(((x-1)^(1/3)-1)^3,x=0)`, and the Mathematica command is `Limit[((x-1)^(1/3)-1)^3,x->0]`.

Solution. By continuity, the limit equals $((-1)^{1/3}-1)^3$. There is more than one possible answer because $(-1)^{1/3}$ has three values in the field of complex numbers.

One value of $(-1)^{1/3}$ is -1; the limit then equals $(-1-1)^3$, or -8. Another value of $(-1)^{1/3}$ is $e^{\pi i/3}$ or $\frac{1}{2}(1+i\sqrt{3})$; the limit then equals $(\frac{1}{2}+\frac{1}{2}i\sqrt{3}-1)^3 = (-\frac{1}{2}+\frac{1}{2}i\sqrt{3})^3 = (e^{2\pi i/3})^3 = e^{2\pi i} = 1$. The third value of $(-1)^{1/3}$ is $e^{-\pi i/3}$; the corresponding limit is the complex conjugate of the preceding case, namely 1 again.

Remark. One student reported that his TI-89 calculator gives the same answer as Maple and Mathematica do. My TI-89, when asked for a root of a real number, returns a real answer if possible and otherwise an error message. I can trick my calculator into giving a principal complex root of a negative real number by declaring explicitly that the problem involves complex numbers. Indeed, my TI-89 says that

$$\lim_{x\to 0} \left((x-1+0i)^{1/3} - 1\right)^3 = 1.$$

Maple can be tricked into giving a root different from the principal complex root as follows: the command `limit((surd(-1,3)-1)^3,x=0)` returns the answer -8.

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2. You know very well that

$$\sin^2(x) + \cos^2(x) = 1 \qquad \text{for every real number } x.$$

Prove that

$$\sin^2(z) + \cos^2(z) = 1 \qquad \text{for every complex number } z.$$

Solution. Method 1. The brute force method is to use the definitions of the trigonometric functions in terms of the exponential function:

$$\begin{aligned}\sin^2(z) + \cos^2(z) &= \left(\tfrac{1}{2i}(e^{iz} - e^{-iz})\right)^2 + \left(\tfrac{1}{2}(e^{iz} + e^{-iz})\right)^2 \\ &= -\tfrac{1}{4}(e^{2iz} - 2 + e^{-2iz}) + \tfrac{1}{4}(e^{2iz} + 2 + e^{-2iz}) \\ &= 1.\end{aligned}$$

This method is probably what you used to solve the corresponding homework exercise 20 on page 54 of the textbook.

Method 2. The entire function $\sin^2(z) + \cos^2(z)$ has a Maclaurin series expansion. You know that the coefficients in the expansion can be expressed in terms of derivatives of the function evaluated at the origin. Since the Maclaurin series coefficients of the real-valued function $\sin^2(x) + \cos^2(x)$ are expressed in exactly the same way, the series expansion for the complex-valued function is the same as the series expansion for the real-valued function, except that the variable x is renamed as z. Therefore knowing that the series for the real-valued function reduces to its constant term implies the same conclusion about the complex-valued function. This idea is sometimes called *the principle of persistence of functional relations*.

Method 3. Consider the derivative:

$$\frac{d}{dz}\left(\sin^2(z) + \cos^2(z)\right) = 2\sin(z)\cos(z) + 2\cos(z)(-\sin(z)) = 0.$$

Since the derivative of the entire function $\sin^2(z)+\cos^2(z)$ is identically equal to 0, the function reduces to a constant function. Evaluating at the origin shows that the value of the constant is 1.

3. Do **either** part (a) **or** part (b).

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(a) Determine a (non-closed) path γ in the complex plane such that

$$\int_\gamma (2z+1)\,dz = -1.$$

Solution. Since the integrand is analytic in the entire plane, the value of the integral depends only on the endpoints of the path. If the path γ joins the complex number a to the complex number b, then the value of the integral is $[z^2+z]_a^b$, or b^2+b-a^2-a. There are lots of choices of a and b for which $b^2+b-a^2-a=-1$. For example, $a=0$ and $b=\frac{1}{2}(-1\pm i\sqrt{3}\,)$; or $a=-\frac{3}{2}$ and $b=-\frac{1}{2}$.

(b) The value of the line integral $\displaystyle\int_\gamma \frac{1}{z^2(z^2+1)}\,dz$ depends on γ, the integration path. What are the possible values of this integral as γ varies over all simple closed curves?

Solution. If the curve passes through one of the singular points (0, i, or $-i$), then the integral is divergent. Otherwise, the integral equals $\pm 2\pi i$ times the sum of the residues of the integrand at the poles that are inside the curve (take the $+$ sign if the curve is oriented in the standard counterclockwise direction but the $-$ sign if the orientation is clockwise).

The residue of the integrand at the simple pole at i equals $i/2$, and the residue at the simple pole at $-i$ equals $-i/2$. The residue at the double pole at 0 equals 0, as can be seen without any calculation by observing that the integrand is an even function, so the coefficient of the $1/z$ term in the Laurent series has to be 0.

The sum of the residues at the poles inside the curve can be either 0 or $i/2$ or $-i/2$, so the value of the integral is either 0 or $-\pi$ or π.

4. Find an entire function $f(z)$ whose real part $u(x,y)$ equals x^2-y^2-2y (where, as usual, x and y denote the real part and the imaginary part of the complex variable z).

Solution. **Method 1.** Since $\partial u/\partial x = 2x$, and by the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y$, it follows that $v(x,y) = 2xy + g(x)$

for some function g. Then $\partial v/\partial x = 2y + g'(x)$, and by the Cauchy-Riemann equations $\partial v/\partial x = -\partial u/\partial y = 2y + 2$, so $g'(x) = 2$ and $g(x) = 2x$ (plus an arbitrary real constant). Thus our analytic function is $x^2 - y^2 - 2y + i(2xy + 2x)$ (plus an arbitrary imaginary constant). This expression can be rewritten as $z^2 + 2iz$ (plus an imaginary constant).

Method 2. The derivative $f'(z)$ is a two-dimensional limit, and the existence of this limit means that we get the same limit along every direction. Hence

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

By the Cauchy-Riemann equations, $\partial v/\partial x = -\partial u/\partial y$, so

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x + 2iy + 2i = 2z + 2i.$$

Therefore $f(z) = z^2 + 2iz$ (plus a constant).

Method 3. Use some other trickery. For instance, since $u(x, y)$ is a quadratic polynomial in x and y, the function $f(z)$ must be a quadratic polynomial in z. Try to find complex numbers a, b, and c such that $\operatorname{Re}(az^2 + bz + c) = x^2 - y^2 - 2y$.

5. Give an example of a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ that has radius of convergence equal to 3 and that represents an analytic function having no zeroes.

   Solution. One solution is the geometric series $\sum_{n=0}^{\infty} \frac{1}{3^n} z^n$, which converges when $|z| < 3$ and diverges when $|z| \geq 3$. The sum of the series inside the disc of radius 3 is $(1 - \frac{z}{3})^{-1}$, which is never equal to 0.

6. Evaluate the integral

$$\int_{|z|=1} z^{407} \cos(1/z)\, dz,$$

   where the integration curve is the unit circle with its usual counterclockwise orientation. (Recall that $\sum_{n=0}^{\infty} (-1)^n w^{2n}/(2n)! = \cos(w)$.)

Solution. To apply the residue theorem, we need to find the coefficient of the $1/z$ term in the Laurent series expansion of the integrand, which is to say the $1/z^{408}$ term in the Laurent series expansion of $\cos(1/z)$. The coefficient is $+1/408!$, so the value of the integral is $2\pi i/408!$. This complex number has absolute value about 10^{-889}, so your calculator thinks it is zero, but it is not precisely zero!

7. How many solutions are there to the equation

$$z^4 + 4 = e^{-z}$$

in the right-hand half-plane where $\operatorname{Re}(z) > 0$? How do you know?

Solution. Apply Rouché's theorem on a large semi-circular contour that follows the arc $Re^{i\theta}$ for θ from $-\pi/2$ to $\pi/2$ and then the imaginary axis from iR to $-iR$. The key point is that $|e^{-z}| \le 1$ when $\operatorname{Re}(z) \ge 0$. On the other hand, $|z^4 + 4| \ge 4 > 1$ when z is on the imaginary axis, while on the arc, $|z^4 + 4| \ge 12 > 1$ if $R \ge 2$. Therefore the equations $z^4 + 4 = 0$ and $z^4 + 4 - e^{-z} = 0$ have the same number of solutions inside the semi-circle. The first equation evidently has two solutions inside the semi-circle (namely, $4^{1/4}e^{i\pi/4}$ and $4^{1/4}e^{-i\pi/4}$), so the second equation also has two solutions. Since nothing changes in this argument as $R \to \infty$, our equation has exactly two solutions in the right-hand half-plane.

Remark. The equation has infinitely many solutions in the left-hand half-plane, but this deduction is harder to make.

8. Do **either** part (a) **or** part (b).

   (a) Either find a one-to-one conformal mapping from the punctured disc $\{ z : 0 < |z| < 1 \}$ onto the annulus $\{ z : 1 < |z| < 2 \}$ or prove that none exists.

   Solution. We argue by contradiction to show that no such mapping f can exist. If f did exist, then since the range is bounded, the isolated singularity of f at the origin would be a removable singularity. Suppose the singularity to have been removed by defining $f(0)$ to be equal to $\lim_{z\to 0} f(z)$; call this value w_0.

Now view f as an analytic function on the open unit disc. By continuity, the point w_0 lies in the closed annulus, but w_0 cannot lie on the boundary of the annulus, because a nonconstant analytic function maps an open set to an open set. Hence w_0 lies in the open annulus, so there is a point z_0 in the punctured disc such that $f(z_0) = w_0$.

Now f maps a small neighborhood of z_0 onto a neighborhood of w_0, but f also maps a small neighborhood of 0 onto a neighborhood of w_0. Hence all points near w_0 are double covered by f, so f cannot be one-to-one after all.

(b) Either find a one-to-one conformal mapping from the first quadrant $\{ z : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0 \}$ onto the strip $\{ z : |\operatorname{Im}(z)| < 1 \}$ or prove that none exists.

Solution. The Riemann mapping theorem implies that such a mapping exists. To write one down, observe that the principal branch of $w = \log(z)$ maps the first quadrant of the z-plane to the strip in the w-plane where $0 < \operatorname{Im}(w) < \pi/2$. Dilating the image by the factor $4/\pi$ adjusts the strip to width 2, and subtracting i centers the strip on the horizontal axis. Thus the desired mapping is given by the function $-i + \frac{4}{\pi}\log(z)$.

9. For the function $\dfrac{1+z}{z(1-z)}$, find a Laurent series in powers of z and $\dfrac{1}{z}$ that converges when $0 < |z| < 1$.

Solution. Using partial fractions and the geometric-series expansion, you can write

$$\frac{1+z}{z(1-z)} = \frac{1}{z} + \frac{2}{1-z} = \frac{1}{z} + \sum_{n=0}^{\infty} 2z^n.$$

10. The TI-89 calculator, Maple, and Mathematica all agree that

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi\sqrt{2}}{4}.$$

Use contour integration and residues to prove this formula.

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Solution. Method 1. Consider $\int_\gamma \frac{z^2}{z^4+1}\,dz$, where γ is a semi-circular contour consisting of the real axis from $-R$ to R and the arc $Re^{i\theta}$ for θ from 0 to π. The integrand has simple poles in the upper half-plane at $e^{\pi i/4}$ and $e^{3\pi i/4}$. If $R > 1$, then the residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{z^2}{z^4+1}\,dz &= 2\pi i\left(\frac{z^2}{4z^3}\bigg|_{z=e^{\pi i/4}} + \frac{z^2}{4z^3}\bigg|_{z=e^{3\pi i/4}}\right)\\ &= \frac{2\pi i}{4}\left(e^{-\pi i/4}+e^{-3\pi i/4}\right)\\ &= \frac{\pi i}{2}\left(\frac{1-i}{\sqrt{2}}+\frac{-1-i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}\,.\end{aligned}$$

The integral over the arc in the upper half-plane is bounded in modulus by $\frac{R^2}{R^4-1}\cdot \pi R$, which tends to 0 as R tends to ∞. The integral over the real axis equals $\int_{-R}^{R} \frac{x^2}{x^4+1}\,dx$ or, by symmetry, $2\int_0^R \frac{x^2}{x^4+1}\,dx$. Hence

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{1}{2}\cdot\frac{\pi}{\sqrt{2}} = \frac{\pi\sqrt{2}}{4}.$$

Method 2. Consider $\int_\gamma \frac{z^2}{z^4+1}\,dz$, where γ consists of the real axis from 0 to R, the arc $Re^{i\theta}$ for θ from 0 to $\pi/2$, and the imaginary axis from iR to 0. There is only one pole inside the contour, so

$$\int_\gamma \frac{z^2}{z^4+1}\,dz = 2\pi i\cdot\frac{z^2}{4z^3}\bigg|_{z=e^{i\pi/4}} = \frac{2\pi i}{4}e^{-i\pi/4} = \frac{\pi}{2}\cdot\frac{1+i}{\sqrt{2}}\,.$$

As above, the contribution from the arc tends to 0 when $R\to\infty$, and the integral over the real axis equals $\int_0^R \frac{x^2}{x^4+1}\,dx$. The contribution from the imaginary axis equals $\int_R^0 \frac{(iy)^2}{(iy)^4+1}\,i\,dy$ or $i\int_0^R \frac{x^2}{x^4+1}\,dx$. Thus

$$(1+i)\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2}\cdot\frac{1+i}{\sqrt{2}}\,,$$

which gives the same answer as before.

Remark. In principle, this integral can be done by techniques from second-semester calculus, but the computation would be lengthy.