Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Determine the residue of the function $\dfrac{4z^2}{z^9 - 1}$ at the simple pole where $z = 1$.

   Solution. Using the formula that the residue of g/h at a simple zero z_0 of h equals $g(z_0)/h'(z_0)$ gives the value

   $$\left.\frac{4z^2}{9z^8}\right|_{z=1}, \qquad \text{which equals} \qquad \frac{4}{9}.$$

   Alternatively, the residue could be computed as

   $$\lim_{z \to 1} \left((z-1) \cdot \frac{4z^2}{z^9 - 1} \right).$$

2. The function $\dfrac{1}{z(1-z)}$ is analytic in the punctured unit disc (where $0 < |z| < 1$). Determine the Laurent series for this function (in powers of z and $1/z$) that converges in this punctured disc.

   Solution. Since $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots$ when $|z| < 1$ (the geometric series formula),

   $$\frac{1}{z(1-z)} = \frac{1}{z} \cdot (1 + z + z^2 + \cdots) = \frac{1}{z} + 1 + z + \cdots = \sum_{n=-1}^{\infty} z^n.$$

   Alternatively, you could use the method of partial fractions to write

   $$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

   (the coefficients of the partial fractions decomposition happen to be particularly simple in this example) and then apply the geometric series formula to the second summand.

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3. Evaluate the complex line integral

$$\int_{|z|=1} \frac{\cos(z)}{\sin(z)}\, dz,$$

where the integration path is the unit circle oriented in the standard counterclockwise direction.

Solution. The integrand is analytic inside the curve except for a simple pole where $z = 0$, and the integral equals $2\pi i$ times the residue of the integrand at that pole. The residue equals

$$\left. \frac{\cos(z)}{\frac{d}{dz}\sin(z)} \right|_{z=0}, \qquad \text{namely } 1,$$

so the value of the integral is $2\pi i$.