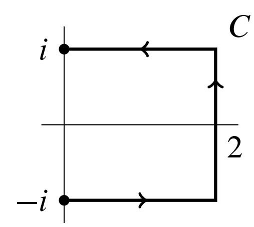
Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:
   (a) Cauchy's integral formula;
   (b) the ratio test for convergence of series of complex numbers.

2. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{\cos(3z)}{z^3}\,dz$ when C is the unit circle (that is, the set of points z for which $|z| = 1$) oriented in the usual counterclockwise direction.



3. Evaluate the integral $\int_C \frac{1}{z^2}\,dz$ when C is the indicated path that goes from the point $-i$ to the point i along three sides of a square.

4. Determine all values of the real number b for which the series $\sum_{n=1}^{\infty} \frac{b^n + i^n}{(b+i)^n}$ converges.

5. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2+\cos(n)}{3^n + 4^n} z^n.$$

6. Show that if $\mathrm{Im}(z) \neq 0$, then $\sin(z) \neq 0$. In other words, the only values of z in the complex plane for which $\sin(z)$ can be equal to 0 are the values on the real axis where the real sine function is equal to 0.

Extra credit

Viewing e^z as a transformation from the z-plane to the w-plane, find a region in the z-plane on which the function e^z is a one-to-one transformation onto the upper half-plane (the set of points w having positive imaginary part).