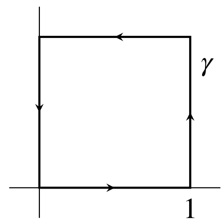
Examination 2

Instructions Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Let γ denote the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, oriented counterclockwise as usual. (See the figure.)

   Determine the value of the line integral $\int_\gamma \operatorname{Re}(z)\, dz$.

2. Suppose $v(x, y) = x^3 - 3xy^2 - 4y$. Determine a function $u(x, y)$ such that $u + iv$ is an analytic function.

3. Let γ denote a simple closed curve, oriented counterclockwise, and suppose $f(z) = \dfrac{z}{z^2 - 1}$. What are the possible values of the integral $\int_\gamma f(z)\, dz$ for different choices of the curve γ?

4. If n is a natural number, and
   $$\int_{|z|=1} \frac{\cos(z)}{z^n}\, dz = 0,$$
   then what can you deduce about the number n?

5. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\cos(in)}{2^n + 3^n} \right) z^n$.

6. Give an example of a function $f(z)$ whose Taylor series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(z-4)^n$ with center at the point 4 has radius of convergence equal to 2.

Extra Credit

Lee and Orville conjecture that if f is an entire function such that $|f(z)| \le \sqrt{|z|}$ for every z, then f must be a constant function.

Lee says, "The only plausible candidate for $f(z)$ is $z^{1/2}$, but this function is not entire: the derivative does not exist when $z = 0$. So I think that the conjecture must be true."

Orville says, "Certainly f cannot be a nonconstant *polynomial*, for then $|f(z)|$ would grow more or less like $|z|^n$ for some positive integer n, which is faster growth than $|z|^{1/2}$. But I am not sure about general entire functions, that is, power series with infinite radius of convergence."

What do you think? Can you prove Lee–Orville's theorem, or can you find a counterexample?