Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. Simplify the complex expression $\dfrac{(1+i)^{12}}{(1-i)^9}$ to the standard form $x + yi$, where x and y are real numbers.

2. If $z = x + iy$ and $f(z) = (\cos(x) + i\sin(x))e^y$, is f an analytic function? Explain.

3. Evaluate the complex line integral $\displaystyle\int_C \frac{1}{z}\,dz$ when C is the line segment in the complex plane that starts at the point 1 and ends at the point i.

4. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{i^n + 2}{(3i+4)^n} z^n$.

5. Let C denote the circle of radius 1 centered at 0, equipped with the usual counterclockwise orientation. Evaluate
$$\int_C \frac{\cos(z) + e^z}{z^{407}}\,dz.$$

6. Give an example of a Laurent series (in powers of z and $1/z$) that converges in the annulus where $3 < |z| < 4$ and in no larger annulus.

7. Describe the image of the unit square in the complex plane (the set of all z for which $0 \le \operatorname{Re}(z) \le 1$ and $0 \le \operatorname{Im}(z) \le 1$) under the mapping that sends z to z^2.

Extra credit

Use the residue theorem to show that $\displaystyle\int_0^\infty \frac{x^2}{1+x^5}\,dx = \frac{\pi/5}{\sin(2\pi/5)}$.

[This problem appeared on the qualifying examination for PhD students in the Department of Mathematics in August.]