Recap of definitions from last time

- A function $f(z)$ is *analytic* on an open set if $f'(z)$ exists at every point of the set.
- A function $u(x, y)$ is *harmonic* on an open set if $u_{xx} + u_{yy} = 0$ (Laplace's equation) at every point of the set.
- If u is a harmonic function, then v is a *harmonic conjugate* of u if the function $u + iv$ is analytic.

Quiz

1. (a) Write the Cauchy–Riemann equations.
   (b) Give a concrete example of an analytic function.
   (c) Give an example of a function that is not analytic.

2. (a) Write Laplace's equation.
   (b) Give a concrete example of a harmonic function.
   (c) Give an example of a function that is not harmonic.

Reminders on real linear approximation

A linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ is represented by matrix multiplication:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The transformation is *invertible* when the determinant of the matrix is nonzero: $ad - bc \neq 0$.

A nonlinear transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ has a *linear approximation* given by the Jacobian matrix:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - \begin{pmatrix} u(0,0) \\ v(0,0) \end{pmatrix} \approx \begin{pmatrix} u_x(0,0) & u_y(0,0) \\ v_x(0,0) & v_y(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The transformation is *locally invertible* if the Jacobian determinant is nonzero.

Jacobian matrix and Cauchy–Riemann equations

Suppose $f(z)$ is an analytic function, and $f = u + iv$. The Jacobian matrix of the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$ can be rewritten:

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}.$$

Then the Jacobian determinant equals $u_x^2 + v_x^2$, or $|u_x + iv_x|^2$, or $|f_x|^2$, or $|f'|^2$.

Conclusion: The transformation $z \mapsto f(z)$ is locally invertible when $f'(z) \neq 0$.

Assignment

- Section II.3, Exercise 1(b)
- Section II.4, Exercise 2
- Section II.5, Exercise 1(f)