Reminder

Exam 2 takes place on Thursday, October 26.

Refresher from last week

- If $f = u + iv$, and f is a reasonable function, then f is analytic if and only if the functions u and v satisfy the Cauchy–Riemann equations.

  Here "reasonable" means that $\partial f/\partial x$ and $\partial f/\partial y$ are continuous functions.

- If u is a reasonable function, then u is harmonic if and only if $u_{xx} + u_{yy} = 0$.

  Here "reasonable" means that the partial derivatives u_{xx}, u_{yy}, u_{xy}, and u_{yx} are continuous functions.

- If $f = u + iv$, and f is analytic, then u is harmonic.
- A function v is a harmonic conjugate of u if $u + iv$ is an analytic function.

A subtlety

If u is harmonic, must there be an analytic function f of which u is the real part?

Equivalent question: If u is harmonic, must there be a harmonic conjugate function v?

Answer: It depends on the domain of the function.
If the domain has no holes, then the answer is "yes."
But if the domain has a hole, then the answer is "not necessarily."

Example

On the annulus $\{ z : 1 < |z| < 2 \}$, the function $\ln |z|$ is harmonic. This function looks like it should be the real part of $\log(z)$, but the imaginary part $\arg(z)$ is not a well-defined continuous function on a loop that circles the origin.

Second refresher from last week

If $f = u + iv$, and f is analytic, then the Jacobian determinant of the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$ equals $|f'|^2$.

Deduction: An analytic function is locally invertible at points where the derivative is nonzero.

Example

Suppose $f(z) = e^z$.
Then $f'(z) = e^z$, which is never equal to 0.
So the exponential function is locally invertible.

Any local inverse is a branch of the logarithm function.
The derivative of $\log(z)$ can be computed by the chain rule:

$$z = e^{\log(z)}, \qquad \text{so} \qquad \frac{d}{dz} z = \frac{d}{dz} e^{\log(z)}.$$

Then

$$1 = e^{\log(z)} \frac{d}{dz} \log(z) = z \frac{d}{dz} \log(z),$$

so the derivative of any branch of $\log(z)$ equals $\dfrac{1}{z}$.

Assignment

- Dust off your multi-variable chain rule from real calculus to solve Exercise 5 in Section II.5.

A possible starting point is

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x). \end{aligned}$$