Exam results

- Mean 86
- Median 87
- Maximum 102

Grade computation: $40 + \sum_{j=1}^{6} n_j + C$, where $0 \leq n_j \leq 10$, and $0 \leq C \leq 10$.

Solutions are posted.

Reminders on line integrals from calculus

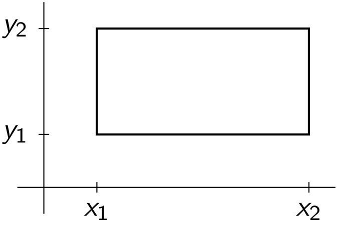
Example

Evaluate the integral $\int_C (xy\,dx + x^2\,dy)$, where C is the part of the parabola $y = x^2$ joining $(0,0)$ to $(1,1)$.

If $y = x^2$, then $dy = 2x\,dx$, so the integral becomes

$$\int_0^1 x^3\,dx + x^2 \cdot 2x\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}.$$

Application of the fundamental theorem of calculus



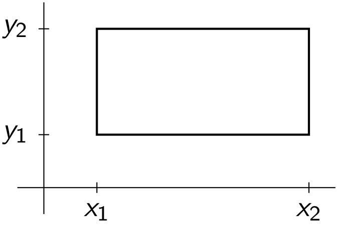
Consider a double integral of a partial derivative:

$$\iint_{\text{rectangle } R} \frac{\partial Q}{\partial x}\, dx\, dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial Q}{\partial x}\, dx\, dy$$

$$= \int_{y_1}^{y_2} Q(x_2, y) - Q(x_1, y)\, dy$$

$$= \int_{\text{oriented } \partial R} Q(x, y)\, dy$$

Continuation



Similarly for the other partial derivative:

$$\iint_{\text{rectangle } R} \frac{\partial P}{\partial y} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial P}{\partial y} \, dy \, dx$$

$$= \int_{x_1}^{x_2} P(x, y_2) - P(x, y_1) \, dx$$

$$= -\int_{\text{oriented } \partial R} P(x, y) \, dx$$

Green's theorem

Generalize from a rectangle to an arbitrary simple closed curve:

Theorem (Green's theorem in the plane)

If C is a simple closed curve, oriented counterclockwise, bounding a region G, and if the functions $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on $G \cup C$, then

$$\int_C (P\,dx + Q\,dy) = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Example

Suppose $P = -y$, $Q = x$, and C is the unit circle. Parametrize C by $x(t) = \cos(t)$ and $y(t) = \sin(t)$. The left-hand side is $\int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = 2\pi$. The right-hand side is $\iint 2\,dxdy = 2\pi$. Green's formula checks out in this example.

Green's theorem and analytic functions

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C (u+iv)\,(dx+i\,dy)\\
&= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\\
&= \iint_G \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\,dy\\
&= 0 \qquad \text{if the Cauchy–Riemann equations hold.}
\end{aligned}$$

Theorem (Cauchy's integral theorem)

If $f(z)$ is analytic on and inside a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

Assignment

Exercise 1 parts (a), (b), and (c) in Section III.1.

(All three parts have answers in the back of the book.)