Recap on differentials

1. The differential $P(x,y)\,dx + Q(x,y)\,dy$ is *closed* when $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

2. The differential $P(x,y)\,dx + Q(x,y)\,dy$ is *exact* when there is a function h such that $\frac{\partial h}{\partial x} = P$ and $\frac{\partial h}{\partial y} = Q$.

Path independence of integrals

To say that the integral of a differential $P\,dx + Q\,dy$ is *independent of path* in a domain G means that if C_1 and C_2 are two arbitrary paths in G joining point A to point B, then

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy.$$

Exact differentials have the path independence property, because

$$\int_C dh = h(B) - h(A).$$

Path independence and closed loops

The integral of a differential $P\,dx + Q\,dy$ is independent of path in a domain G precisely when the integral over every *closed* path equals 0.

Why? If paths C_1 and C_2 have the same endpoints, then make a closed path out of C_1 followed by C_2 backward.

Relationship between closed and exact differentials

Exact differentials are closed (by equality of mixed second-order partial derivatives).

Example

In the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$, the differential $d\theta$ is closed but not exact.
The angle θ is not a well-defined continuous function in the annulus, because the angle is determined only up to addition of multiples of 2π. But the derivative of a constant is 0, so $d\theta$ is well defined (independent of the multiple of 2π).

Nonetheless, Green's theorem implies that closed differentials are exact in domains with no holes (*simply connected* domains).

Path deformation principle for analytic functions

Theorem

If C_1 and C_2 are two paths with the same endpoints, and these paths can be deformed into each within the region where f is analytic, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Assignment

- Exercise 2 in Section III.2.
- Exercise 4 in Section IV.1.