Ratio test and root test revisited

Theorem (Ratio test)

First compute $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$.

- *If the limit exists and is less than* 1*, then* $\sum_{n=0}^{\infty} c_n$ *converges.*
- *If the limit exists and is greater than* 1 *(possibly* $+\infty$*), then* $\sum_{n=0}^{\infty} c_n$ *diverges.*
- *If the limit exists and equals* 1*, no deduction can be made.*
- *If the limit does not exist, no deduction can be made.*

Theorem (Root test)

First compute $\lim_{n\to\infty} |c_n|^{1/n}$*. The deductions are the same as in the ratio test.*

A tricky example

Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(3+4i)^n}{3^n + (4i)^n}\, z^n$.

Solution. The idea is that when n is large, the denominator is approximately $(4i)^n$, so the series probably has the same radius of convergence as the series $\displaystyle\sum_{n=1}^{\infty} \frac{(3+4i)^n}{(4i)^n}\, z^n$. The new series can be analyzed by the root test:

$$\lim_{n\to\infty} \left| \frac{(3+4i)^n}{(4i)^n}\, z^n \right|^{1/n} = \lim_{n\to\infty} \frac{5}{4}|z| = \frac{5}{4}|z| \overset{?}{<} 1.$$

The new series therefore has radius of convergence equal to $\dfrac{4}{5}$.

Continuation

How to justify that the approximation method is valid?

Observe that

$$\frac{1}{4} \cdot 4^n \le 4^n - 3^n \le |3^n + (4i)^n| \le 3^n + 4^n < 2 \cdot 4^n.$$

Therefore

$$\left| \frac{(3+4i)^n}{2 \cdot 4^n} \, z^n \right|^{1/n} \le \left| \frac{(3+4i)^n}{3^n + (4i)^n} \, z^n \right|^{1/n} \le \left| \frac{(3+4i)^n}{\frac{1}{4} \cdot 4^n} \, z^n \right|^{1/n}$$

Since $\lim_{n\to\infty} 2^{1/n} = 1$ and $\lim_{n\to\infty} (1/4)^{1/n} = 1$, the squeeze theorem for limits implies that the original series and the approximate series have the same radius of convergence.

Assignment to hand in next time

Some review exercises:

1. When is the final exam?
2. Determine all values of the integer n for which $i^n = 1$.
3. Determine all values of the complex number z for which $e^z = 1$.