Recap: Residues

The residue of $f(z)$ at the point z_0 is the coefficient of $\dfrac{1}{z - z_0}$ in the Laurent series expansion of f in positive and negative powers of $(z - z_0)$ that converges in a punctured disk centered at z_0.

Example

If $f(z) = \exp\left(\dfrac{\cos(z)}{z^2}\right)$, then the residue of f at the point 0 equals 0, since the function is even, so there is no $\dfrac{1}{z}$ term in the Laurent series!

More on residues at simple poles

- If $f(z) = \dfrac{g(z)}{z - z_0}$, where $g(z)$ is analytic at z_0, then the residue of $f(z)$ at z_0 equals $g(z_0)$.
- If $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are analytic, and $h(z_0) = 0$ to *first order*, then the residue of $f(z)$ at z_0 equals $\dfrac{g(z_0)}{h'(z_0)}$.

Example

If $f(z) = \dfrac{e^z}{\sin(z)}$, then f has a singularity when $z = n\pi$ for each integer n, and

$$\mathrm{Res}(f, n\pi) = \frac{e^{n\pi}}{\cos(n\pi)} = (-1)^n e^{n\pi}.$$

Recap: Residue theorem

Theorem

When f is analytic on and inside a simple closed curve C except for some isolated singularities, then $\oint_C f(z)\,dz$ equals $2\pi i$ times the sum of the residues of f at the singular points inside C.

Example of applying the residue theorem

Compute $\displaystyle\int_0^{2\pi} \frac{1}{5+3\cos(\theta)}\,d\theta$.

Solution:

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{5+3\cos(\theta)}\,d\theta &= \int_0^{2\pi} \frac{1}{5+\frac{3}{2}(e^{i\theta}+e^{-i\theta})}\,d\theta \\
&\overset{(z=e^{i\theta})}{=} \int_{|z|=1} \frac{1}{5+\frac{3}{2}(z+\frac{1}{z})}\cdot\frac{dz}{iz} \\
&\overset{\text{(algebra)}}{=} \frac{2}{i}\int_{|z|=1} \frac{1}{3z^2+10z+3}\,dz \\
&\overset{\text{(factor)}}{=} \frac{2}{i}\int_{|z|=1} \frac{1}{(3z+1)(z+3)}\,dz \\
&\overset{\text{(res. thm.)}}{=} \frac{2}{i}\times 2\pi i\times(\text{residue at } -1/3) \\
&= \frac{2}{i}\times 2\pi i\times\frac{1}{8} = \frac{\pi}{2}
\end{aligned}$$

Quiz

1. Evaluate the integral $$\int_{|z|=409} \frac{z}{z^2+1}\,dz.$$

2. Suppose $f(z) = \dfrac{\cos(\pi z)}{?}$ with an unknown denominator.

   If $\mathrm{Res}(f,0) = 1$ and $\mathrm{Res}(f,1) = -1$, then what could the denominator of f be?