Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Suppose $f(z) = \left(\overline{z}\right)^2$ for every z. Show that the complex derivative $f'(0)$ exists and equals 0. (Recall that the notation $\overline{z}$ means the complex conjugate of z.)

   Solution. To determine $f'(0)$, consider the following limit:

   $$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\left(\overline{0+h}\right)^2 - \left(\overline{0}\right)^2}{h} = \lim_{h\to 0} \frac{\left(\overline{h}\right)^2}{h}.$$

   Write h in polar form as $re^{i\theta}$. Then $\overline{h} = re^{-i\theta}$, and the preceding limit can be rewritten as follows:

   $$\lim_{r\to 0} \frac{r^2 e^{-2i\theta}}{re^{i\theta}} = e^{-3i\theta} \lim_{r\to 0} r.$$

   This limit evidently exists and equals 0.

   An alternative way to handle the limit $\lim\limits_{h\to 0} (\overline{h})^2/h$ is to observe that

   $$0 \le \left| \frac{(\overline{h})^2}{h} \right| = |h|.$$

   Since $\lim\limits_{h\to 0} |h| = 0$, the sandwich theorem implies that $\lim\limits_{h\to 0} (\overline{h})^2/h$ exists and equals 0.

2. Determine values of the real numbers a, b, and c to make the function

   $$x^2 + ay^2 + y + i(bxy + cx)$$

   be an analytic function of the complex variable $x + yi$.

   Solution. Method 1. Apply the Cauchy–Riemann equations, setting $u(x, y)$ equal to the expression $x^2 + ay^2 + y$ and $v(x, y)$ equal to $bxy + cx$. The first Cauchy–Riemann equation implies that

   $$2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = bx, \qquad \text{so} \qquad b = 2.$$

   The second Cauchy–Riemann equation implies that

   $$2ay + 1 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -(by + c) = -(2y + c), \qquad \text{so} \qquad a = -1 \quad \text{and} \quad c = -1.$$

   Method 2. Use the "guess and check" technique. Since the given expression is a quadratic polynomial in x and y, the underlying analytic function must be a quadratic polynomial in $x + yi$. In other words, there must be complex numbers A and B such that

   $$\begin{aligned} x^2 + ay^2 + y + i(bxy + cx) &= A(x + yi)^2 + B(x + yi) \\ &= A(x^2 - y^2) + 2Aixy + Bx + Biy. \end{aligned}$$

Matching the x^2 terms shows that $A = 1$, and then matching the y^2 terms shows that $a = -1$. Matching the xy terms shows that $b = 2$. Matching the y terms shows that $B = -i$, and then matching the x terms shows that $c = -1$.

3. If $u(x, y) = 4x^3y - 4xy^3$, is there a function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is an analytic function? Explain.

Solution. Method 1. Apply the Cauchy–Riemann equations to try to find v:

$$12x^2y - 4y^3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

so $v(x, y) = 6x^2y^2 - y^4 + c(x)$ for some function $c(x)$. The second Cauchy–Riemann equation implies that

$$4x^3 - 12xy^2 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\left(12xy^2 + c'(x)\right).$$

Comparing the two sides of this equation reveals that $c'(x) = -4x^3$, so $c(x) = -x^4 + k$ for some constant k. Thus $v(x, y) = 6x^2y^2 - y^4 - x^4 + k$.

Since the Cauchy–Riemann equations hold, and $u+iv$ is a reasonable function (a polynomial is a reasonable function!), this function is analytic. A little more work reveals that the underlying analytic function can be written as $i(-z^4 + k)$.

Method 2. Compute the Laplacian of the function u:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 24xy - 24xy = 0.$$

Thus u is a harmonic function on the entire complex plane. By a general theorem, there is a harmonic conjugate function v (since the plane is a region without holes). Thus you can justify the answer, "yes," without actually computing v.

4. The complex tangent and secant functions are defined by analogy with the real counterparts: $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$ and $\sec(z) = \dfrac{1}{\cos(z)}$. Is it correct to say that $\tan(z)$ is an analytic function having derivative $(\sec(z))^2$? Explain why or why not.

Solution. The quotient rule applies to complex derivatives, so the quotient of two analytic functions is analytic, as long as the denominator is not equal to zero. Therefore $\tan(z)$ is an analytic function on the domain

$$\mathbb{C} \setminus \{(2n+1)\pi/2 : n \text{ is an integer }\}.$$

The derivative is given by the quotient rule:

$$\tan'(z) = \frac{\cos(z)\sin'(z) - \sin(z)\cos'(z)}{(\cos(z))^2} = \frac{\cos(z)\cos(z) - \sin(z)(-\sin(z))}{(\cos(z))^2} = \frac{1}{(\cos(z))^2},$$

since $(\cos(z))^2 + (\sin(z))^2 = 1$. So the complex derivative of the complex tangent function is indeed the square of the complex secant function.

5. Suppose f is an analytic function defined on $\{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$, the upper half-plane. Given the information that

$$f(f(z)) = z \qquad \text{and} \qquad f'(z) = \frac{1}{z^2} \qquad \text{for every } z,$$

find the most general possible expression for $f(z)$.

Solution. Method 1. The information about the derivative shows that $f(z) = \frac{-1}{z} + c$ for some constant c. The first equation puts a constraint on c: namely,

$$z = f(f(z)) = \frac{-1}{-\frac{1}{z} + c} + c = \frac{z}{1 - cz} + c.$$

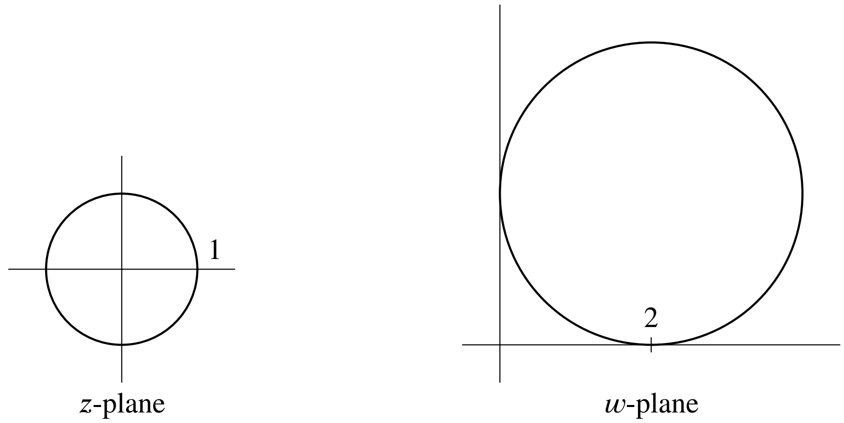
The left-hand side represents the identity transformation, but the right-hand side evidently is not the identity transformation unless $c = 0$. (If $c \neq 0$, then the right-hand side has an undefined point: namely, $z = 1/c$.) The conclusion is that $f(z) = -1/z$.

Method 2. Differentiate the first equation using the chain rule to see that $f'(f(z))f'(z) = 1$. The given information about the derivative then shows that

$$\frac{1}{f(z)^2} \cdot \frac{1}{z^2} = 1, \qquad \text{or} \qquad \frac{1}{z^2} = f(z)^2.$$

Therefore $f(z)$ must be either $+1/z$ or $-1/z$. Both of these expressions obey the property that $f(f(z)) = z$, but only the second one has the correct derivative. Therefore $f(z) = -1/z$.

6. Determine values of the complex numbers a, b, c, and d to ensure that if $w = \dfrac{az+b}{cz+d}$, then the unit circle centered at 0 in the z-plane maps to the circle of radius 2 in the first quadrant of the w-plane tangent to the coordinate axes. See the figure.



Solution. The simplest solution is to realize the transformation geometrically by first translating by $1+i$ and then dilating by a factor of 2. The composition of these two transformations has the form $z \mapsto 2z + 2(1+i)$. This expression matches the required form if

$$c = 0, \qquad d = 1, \qquad a = 2, \qquad \text{and} \qquad b = 2 + 2i.$$

There are infinitely many other solutions. For instance, you could multiply by i (that is, rotate by angle $\pi/2$ in the z-plane) and then compose with the preceding transformation to obtain the transformation $z \mapsto 2iz + 2(1+i)$. Or you could start with the inversion $1/z$, which reflects the unit circle into itself (since $1/e^{i\theta} = e^{-i\theta} = \overline{e^{i\theta}}$); then composing with the first solution gives the transformation $z \mapsto \frac{2}{z} + 2 + 2i$.

Even more complicated solutions are possible. The fractional linear transformation

$$\frac{(2i-2)z - 2 - 4i}{z - 2}$$

is another solution. Indeed, this transformation takes the point 1 to the point $2i+4$, which lies on the image circle; takes the point -1 to the point $2i$, which lies on the image circle; and takes the point $-i$ to the point $(4i+2)/5$, which has distance 2 from the point $2+2i$ and therefore lies on the image circle. The images of three points suffice to determine a Möbius transformation, and such transformations take circles to circles, so the indicated two circles do correspond to each other under the transformation.

Extra Credit Problem. Show that if u is the real part of a function, and v is the imaginary part, then the Cauchy–Riemann equations for u and v take the following form in polar coordinates:

$$r\,\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \text{and} \qquad r\,\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

Solution. Method 1. Let f denote $u + iv$, and apply the definition of the complex derivative:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

Suppose $z = re^{i\theta}$, and consider what happens when $h \to 0$ in the radial direction. If $h = e^{i\theta}\Delta r$, then the limit becomes

$$\lim_{\Delta r\to 0} \frac{f((r+\Delta r)e^{i\theta}) - f(re^{i\theta})}{e^{i\theta}\Delta r}, \qquad \text{or} \qquad e^{-i\theta}\frac{\partial f}{\partial r}.$$

Next suppose $h \to 0$ in the angular direction. If $h = re^{i(\theta+\Delta\theta)} - re^{i\theta}$, or $re^{i\theta}(e^{i\Delta\theta} - 1)$, then the limit becomes

$$\lim_{\Delta\theta\to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{re^{i\theta}(e^{i\Delta\theta} - 1)}, \qquad \text{or} \qquad \frac{e^{-i\theta}}{r} \lim_{\Delta\theta\to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{\Delta\theta} \cdot \frac{\Delta\theta}{e^{i\Delta\theta} - 1}.$$

Since $\frac{\Delta\theta}{e^{i\Delta\theta}-1}$ converges to the reciprocal of the derivative of e^{it} with respect to t when $t = 0$, the whole expression has the limiting value

$$\frac{e^{-i\theta}}{r} \cdot \frac{\partial f}{\partial\theta} \cdot \frac{1}{i}.$$

The Cauchy–Riemann equations arise by demanding that the limit be independent of the direction, that is,

$$e^{-i\theta}\frac{\partial f}{\partial r} = \frac{e^{-i\theta}}{r} \cdot \frac{\partial f}{\partial\theta} \cdot \frac{1}{i}, \qquad \text{or} \qquad r\frac{\partial f}{\partial r} = -i\frac{\partial f}{\partial\theta}.$$

Substitute $u + iv$ for f to see that

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}.$$

The required equations are the real part of this equation and the imaginary part.

Method 2. Apply the chain rule:

$$r\frac{\partial u}{\partial r} = r\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) = r\left(\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)\right).$$

Similarly,

$$\frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial\theta} = \frac{\partial v}{\partial x}(-r\sin(\theta)) + \frac{\partial v}{\partial y}(r\cos(\theta)).$$

These two expressions match when $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. In other words, if the Cauchy–Riemann equations hold in rectangular coordinates, then $r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial\theta}$.

The second desired equation can be derived analogously:

$$r\frac{\partial v}{\partial r} = r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) = r\left(\frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta)\right), \quad \text{and}$$

$$-\frac{\partial u}{\partial \theta} = -\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) = -\left(\frac{\partial u}{\partial x}(-r\sin(\theta)) + \frac{\partial u}{\partial y}(r\cos(\theta))\right).$$

Again the two expressions match up when the Cauchy–Riemann equations hold in rectangular coordinates.

Method 3. The third method minimizes calculation at the expense of a higher level of abstraction.

The Cauchy–Riemann equations can be interpreted as saying that the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

has the special form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

That is, the diagonal elements of the matrix are equal, and the off-diagonal elements are negatives of each other.

If you know how to multiply matrices, then you can easily check that the product of two such matrices has the same form, and the inverse of a matrix of the special form still has the same form. When you compose two mappings, the Jacobian matrix of the composite function is the product of the two Jacobian matrices. The deduction is that if you change coordinates from (x, y) to $(\tilde{x}, \tilde{y})$, and if $\tilde{x} + \tilde{y}i$ is an analytic function of $x + yi$, then the Cauchy–Riemann equations in the new coordinates have exactly the same form as in the standard coordinates: namely,

$$\frac{\partial u}{\partial \tilde{x}} = \frac{\partial v}{\partial \tilde{y}} \qquad \text{and} \qquad \frac{\partial u}{\partial \tilde{y}} = -\frac{\partial v}{\partial \tilde{x}}.$$

This observation does not by itself solve the problem, because $r + i\theta$ is not an analytic function of $x + yi$. But $\ln(r) + i\theta$ is an analytic function (namely, a branch of the complex logarithm), so taking $\tilde{x}$ to be $\ln(r)$ and $\tilde{y}$ to be θ shows that the Cauchy–Riemann equations can be expressed as follows:

$$\frac{\partial u}{\partial \ln(r)} = \frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial \ln(r)}. \tag{*}$$

The chain rule implies that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \ln(r)}\frac{\partial \ln(r)}{\partial r} = \frac{\partial u}{\partial \ln(r)}\frac{1}{r}, \qquad \text{so} \qquad \frac{\partial u}{\partial \ln(r)} = r\frac{\partial u}{\partial r},$$

and similarly $\dfrac{\partial v}{\partial \ln(r)} = r\dfrac{\partial v}{\partial r}$. Accordingly, the equations (*) are equivalent to what was to be proved.