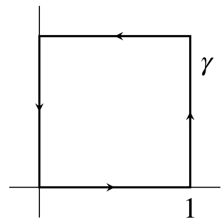
Examination 2

Instructions Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Let γ denote the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, oriented counterclockwise as usual. (See the figure.)



Determine the value of the line integral $\int_\gamma \operatorname{Re}(z)\,dz$.

Solution. Method 1: Parametrize the path. The integral can be expressed as

$$\int_0^1 \operatorname{Re}(t)\,dt + \int_0^1 \operatorname{Re}(1+it)\,i\,dt + \int_0^1 \operatorname{Re}(1-t+i)\,(-1)\,dt + \int_0^1 \operatorname{Re}(i(1-t))\,(-i)\,dt,$$

which simplifies to

$$\int_0^1 (t+i-(1-t)+0)\,dt, \quad \text{or} \quad \int_0^1 (2t-1+i)\,dt, \quad \text{or} \quad i.$$

Method 2: Apply Green's theorem.

$$\int_\gamma x\,dx + x\,i\,dy = \iint_{\text{square}} \left(\frac{\partial(xi)}{\partial x} - \frac{\partial x}{\partial y} \right) dx\,dy = \iint_{\text{square}} (i-0)\,dx\,dy.$$

Since the area of the square is equal to 1, the answer is i.

2. Suppose $v(x,y) = x^3 - 3xy^2 - 4y$. Determine a function $u(x,y)$ such that $u + iv$ is an analytic function.

Solution. Consistency check:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x - 6x = 0.$$

Thus the function v is harmonic in the entire plane, which is a simply connected domain, so there must exist a harmonic function u such that $u + iv$ is harmonic.

To compute u, invoke the Cauchy–Riemann equations and integrate, as follows.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -6xy - 4.$$

Therefore $u(x,y) = -3x^2y - 4x + g(y)$ for some function g. Consequently,

$$-3x^2 + g'(y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -(3x^2 - 3y^2).$$

Comparing the left-hand side with the right-hand side reveals that $g'(y) = 3y^2$, so $g(y) = y^3$ (plus a constant). Therefore $u(x,y) = -3x^2y - 4x + y^3$ (plus a constant).

Remark. The underlying analytic function, $u + iv$, is $iz^3 - 4z$.

3. Let γ denote a simple closed curve, oriented counterclockwise, and suppose $f(z) = \dfrac{z}{z^2-1}$. What are the possible values of the integral $\int_\gamma f(z)\,dz$ for different choices of the curve γ?

Solution. The two singular points of the function f are 1 and -1. If the curve γ encloses neither of these singular points, then Cauchy's theorem implies that the value of the integral is 0. If the curve γ encloses the point 1 but not the point -1, then Cauchy's integral formula implies that

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{z/(z+1)}{z-1}\,dz = 2\pi i \cdot \frac{1}{1+1} = \pi i.$$

If the curve γ encloses the point -1 but not the point 1, then Cauchy's integral formula implies that

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{z/(z-1)}{z+1}\,dz = 2\pi i \cdot \frac{-1}{-1-1} = \pi i.$$

If the curve γ encloses both of the singular points, then the value of the integral is the sum of the preceding two quantities: namely, $2\pi i$. If one of the singular points lies *on* the curve γ, then the integral is not well defined (being a divergent improper integral).

In summary, the possible values of the integral are 0, πi, and $2\pi i$.

Remark. You could alternatively solve the problem by saying that

$$f(z) = \frac{z}{z^2-1} = \frac{1/2}{z-1} + \frac{1/2}{z+1} \qquad \text{(partial fractions)}$$

and then invoking Cauchy's integral formula.

4. If n is a natural number, and

$$\int_{|z|=1} \frac{\cos(z)}{z^n}\,dz = 0,$$

then what can you deduce about the number n?

Solution. Method 1. Cauchy's integral formula for derivatives implies that

$$\int_{|z|=1} \frac{\cos(z)}{z^n}\,dz = \frac{2\pi i}{(n-1)!} \cdot \frac{d^{n-1}}{dz^{n-1}} \cos(z) \Big|_{z=0}.$$

A derivative of $\cos(z)$ of even order equals $\pm\cos(z)$, hence is nonzero when $z = 0$. On the other hand, a derivative of $\cos(z)$ of odd order equals $\pm\sin(z)$, hence is zero when $z = 0$. Thus $n - 1$ must be odd, so n must be even.

Method 2. Expand $\cos(z)$ in a power series and exchange the order of summation and integration to see that

$$\int_{|z|=1} \frac{\cos(z)}{z^n}\,dz = \sum_{k=0}^{\infty} \int_{|z|=1} \frac{(-1)^k}{(2k)!} z^{2k-n}\,dz.$$

The integral of an integer power of z around the unit circle is equal to zero except when the exponent is -1 (in which case the integral is equal to $2\pi i$). Therefore the preceding expression is always equal to zero when n is even, for then $2k - n$ cannot be equal to -1. But if n is odd, then there will be exactly one value of k for which $2k - n = -1$, so the sum will be nonzero.

5. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\cos(in)}{2^n + 3^n}\right) z^n$.

Solution. Method 1. Since $\cos(z)$ is the average of e^{iz} and e^{-iz}, the quantity $\cos(in)$ is the average of e^{-n} and e^n. In particular, the quantity $\cos(in)$ is real and positive. Now $e^{-n} < 1 < e^n$, so

$$\frac{1}{2}e^n < \frac{e^{-n} + e^n}{2} < e^n.$$

On the other hand,

$$3^n < 2^n + 3^n < 2 \cdot 3^n.$$

Therefore

$$\frac{\frac{1}{2}e^n}{2 \cdot 3^n} < \frac{\cos(in)}{2^n + 3^n} < \frac{e^n}{3^n},$$

and

$$\frac{1}{4^{1/n}} \cdot \frac{e}{3} < \left(\frac{\cos(in)}{2^n + 3^n}\right)^{1/n} < \frac{e}{3}.$$

Since $\lim_{n\to\infty} 4^{1/n} = 1$, the squeeze theorem implies that

$$\lim_{n\to\infty} \left(\frac{\cos(in)}{2^n + 3^n}\right)^{1/n} = \frac{e}{3}.$$

By the root test or by the Cauchy–Hadamard formula, the radius of convergence of the given power series is equal to the reciprocal of this value: namely, to $3/e$.

Method 2. By the ratio test, the radius of convergence equals the limit

$$\lim_{n\to\infty} \left| \frac{\cos(in)}{2^n + 3^n} \cdot \frac{2^{n+1} + 3^{n+1}}{\cos(in + i)} \right|$$

if the limit exists. Now

$$\frac{\cos(in)}{\cos(in+i)} = \frac{\cos(in)}{\cos(in)\cos(i) - \sin(in)\sin(i)} = \frac{\cosh(n)}{\cosh(n)\cosh(1) + \sinh(n)\sinh(1)}.$$

Since $\cosh(n)/\sinh(n) \to 1$ when $n \to \infty$, the limit of the preceding expression equals

$$\frac{1}{\cosh(1) + \sinh(1)}, \qquad \text{or} \qquad \frac{1}{e}.$$

On the other hand,

$$\lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} = \lim_{n\to\infty} \frac{2(\frac{2}{3})^n + 3}{(\frac{2}{3})^n + 1} = 3,$$

since $(2/3)^n \to 0$ when $n \to \infty$. Multiplying the two limits together shows that the radius of convergence of the series equals $3/e$.

6. Give an example of a function $f(z)$ whose Taylor series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(z-4)^n$ with center at the point 4 has radius of convergence equal to 2.

Solution. There are many examples. A simple one is $\dfrac{1}{z-2}$. This function is analytic in a disk of radius 2 centered at 4 but is analytic in no larger disk with center 4, so the Taylor series with center 4 must have radius of convergence equal to 2.

Alternatively, you could produce an example by starting with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z-4}{2}\right)^n,$$

which converges precisely when $|z-4|/2 < 1$ and thus has radius of convergence equal to 2. A geometric series sums to the first term divided by 1 minus the ratio, so the underlying analytic function $f(z)$ is

$$\frac{1}{1 - \frac{z-4}{2}}, \qquad \text{or} \qquad \frac{2}{6-z}.$$

Extra Credit

Lee and Orville conjecture that if f is an entire function such that $|f(z)| \le \sqrt{|z|}$ for every z, then f must be a constant function.

Lee says, "The only plausible candidate for $f(z)$ is $z^{1/2}$, but this function is not entire: the derivative does not exist when $z = 0$. So I think that the conjecture must be true."

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Orville says, "Certainly f cannot be a nonconstant *polynomial*, for then $|f(z)|$ would grow more or less like $|z|^n$ for some positive integer n, which is faster growth than $|z|^{1/2}$. But I am not sure about general entire functions, that is, power series with infinite radius of convergence."

What do you think? Can you prove Lee–Orville's theorem, or can you find a counterexample?

Solution. The conjecture is a correct variation of Liouville's theorem.

Method 1. The hypothesis implies, in particular, that $|f(0)| \le 0$, that is, $f(0) = 0$. Therefore the power series expansion of $f(z)$ is divisible by z, so there is an entire function g such that $f(z) = zg(z)$ for every z. The hypothesis implies moreover that

$$|g(z)| = \frac{|f(z)|}{|z|} \le \frac{\sqrt{|z|}}{|z|} = \frac{1}{\sqrt{|z|}} \qquad \text{when } z \ne 0.$$

Therefore $|g(z)| < 1$ when $|z| > 1$. But g is a continuous function, so $|g(z)|$ attains a finite maximum on the closed disk where $|z| \le 1$. Accordingly, the entire function g is bounded in the whole plane. By Liouville's theorem, the function g reduces to a constant C.

Then $f(z) = Cz$. But now the hypothesis implies that $|Cz| \le \sqrt{|z|}$, or $|C|\sqrt{|z|} \le 1$, and this inequality cannot hold for large values of $|z|$ unless $C = 0$. Therefore the function f not only is constant but actually is the constant 0.

Method 2. Adapt the proof of Liouville's theorem that I gave in class. If z_0 is an arbitrary point in the plane, and $R > |z_0|$, then Cauchy's integral formula implies that

$$f(z_0) - f(0) = \frac{1}{2\pi i}\int_{|w|=R} \left(\frac{f(w)}{w - z_0} - \frac{f(w)}{w - 0}\right) dw = \frac{1}{2\pi i}\int_{|w|=R} \frac{z_0 f(w)}{w(w - z_0)}\, dw.$$

Parametrize the integration curve as $Re^{i\theta}$, use that the absolute value of an integral is at most the integral of the absolute value, and bring in the hypothesis to deduce that

$$|f(z_0) - f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|z_0|\sqrt{R}}{R(R - |z_0|)}\, R\, d\theta = \frac{|z_0|\sqrt{R}}{R - |z_0|}.$$

Let R tend to infinity to conclude that $|f(z_0) - f(0)| \le 0$, that is, $f(z_0) = f(0)$. Since the point z_0 is arbitrary, the given function is equal to the constant value $f(0)$.

Method 3. Adapt the proof of Liouville's theorem given in the textbook. Cauchy's formula for the first derivative says that

$$f'(z_0) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{(w - z_0)^2}\, dw \qquad \text{when } R > |z_0|.$$

Bound the absolute value of the integral by a strategy similar to the one used in Method 2:

$$|f'(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{R}}{(R - |z_0|)^2}\, R\, d\theta = \frac{R^{3/2}}{(R - |z_0|)^2}.$$

Holding z_0 fixed, let R tend to infinity to conclude that $f'(z_0) = 0$. But the point z_0 is arbitrary, so the derivative f' is identically equal to zero. Therefore f is a constant function.

Remark. This problem is related to Exercise 4 on page 119 in Section IV.5 of the textbook.