Advanced Calculus I

Instructions Please write your solutions on your own paper.

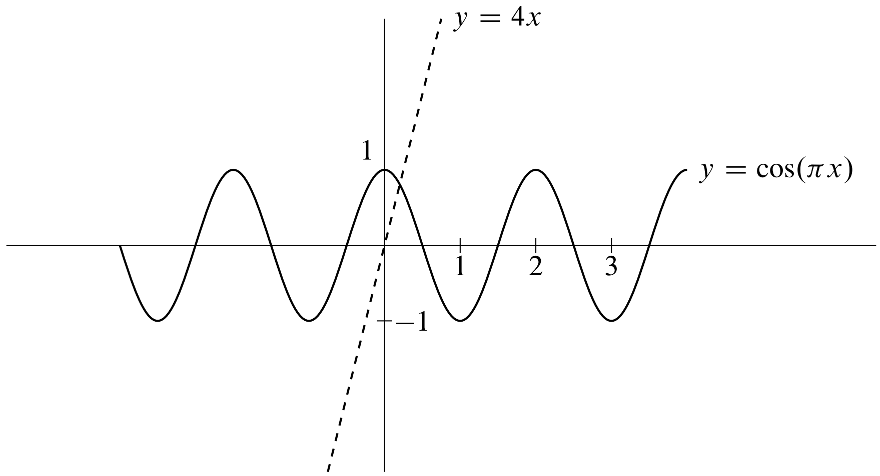
These problems should be treated as essay questions. A problem that says "give an example" or "determine" requires a supporting explanation. In all problems, you should explain your reasoning in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 (the honors section) should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 200 and Section 501

1. The diagram below provides convincing evidence that there is exactly one solution in the real numbers to the equation $\cos(\pi x) = 4x$. But a picture is not a proof.



Your task is to supply a proof, as follows.

a) Apply the intermediate-value theorem to prove that there is at least one real number x between 0 and 1 such that $\cos(\pi x) - 4x = 0$.

Solution. The function $\cos(\pi x) - 4x$ is continuous; when $x = 0$ the value of the function is $\cos(0) - 0$ or 1; and when $x = 1$ the value of the function is $\cos(\pi) - 4$ or -5. By the intermediate-value theorem, the function takes all values between -5 and 1 on the interval $(0, 1)$. In particular, the function takes the value 0.

b) Apply Rolle's theorem (or the mean-value theorem) to prove that there cannot be two distinct real numbers for which $\cos(\pi x) - 4x = 0$.

Solution. The derivative of $\cos(\pi x) - 4x$ equals $-\pi \sin(\pi x) - 4$, and $|-\pi \sin(\pi x)| \le \pi < 4$, so the derivative is never equal to 0. By (the contrapositive of) Rolle's theorem, the function $\cos(\pi x) - 4x$ is one-to-one. In particular, there cannot be two values of x for which $\cos(\pi x) - 4x = 0$.

2. Suppose

$$f(x) = \begin{cases} \log(\cos(\sin(x))), & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

Is the function f continuous at the point where $x = 0$? Explain why or why not.
(You may assume that the logarithm function and the trigonometric functions are continuous on their natural domains.)

Solution. To prove that the function f is continuous at 0, what needs to be shown is that $\lim_{x\to 0} f(x) = f(0)$. Since continuous functions preserve limits,

$$\begin{aligned} \lim_{x\to 0} \log(\cos(\sin(x))) &= \log(\cos(\lim_{x\to 0} \sin(x))) = \log(\cos(\sin(0))) \\ &= \log(\cos(0)) = \log(1) = 0 = f(0). \end{aligned}$$

Thus f is continuous at 0.

3. Suppose a is a positive real number, and

$$f_a(x) = \begin{cases} \dfrac{\sin(x) - x\cos(x)}{|x|^a}, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

Show that f_a is differentiable at 0 when $a \leq 2$.

Solution. What needs to be studied is

$$\lim_{x\to 0} \frac{f_a(x) - f_a(0)}{x - 0} \qquad \text{or} \qquad \lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x\,|x|^a}. \tag{1}$$

Method 1 When $a = 2$, apply l'Hôpital's rule to evaluate the limit (1) as follows:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3} &= \lim_{x\to 0} \frac{\cos(x) - \cos(x) + x\sin(x)}{3x^2} = \lim_{x\to 0} \frac{x\sin(x)}{3x^2} \\ &= \lim_{x\to 0} \frac{\sin(x)}{3x} = \lim_{x\to 0} \frac{\cos(x)}{3} = \frac{1}{3}. \end{aligned}$$

Therefore $f_2'(0) = 1/3$.
When $a < 2$, use that the limit of a product is the product of the limits (if both limits exist):

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x\,|x|^a} &= \lim_{x\to 0} |x|^{2-a} \cdot \frac{\sin(x) - x\cos(x)}{x^3} \\ &= \lim_{x\to 0} |x|^{2-a} \cdot \lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3} \\ &= 0 \cdot \frac{1}{3} = 0. \end{aligned}$$

Therefore $f_a'(0)$ exists and equals 0 when $a < 2$.

Method 2 Approximate the numerator by a Taylor polynomial. Since the successive derivatives of $\sin(x)$ are $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$, ..., it follows that

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{\cos(c_1)}{5!}x^5 \qquad \text{for some } c_1.$$

Similarly,

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{\cos(c_2)}{4!}x^4 \qquad \text{for some } c_2.$$

Therefore $\sin(x) - x\cos(x) = \frac{1}{3}x^3 + \mathcal{E}$, where $|\mathcal{E}| \le |x|^5(\frac{1}{4!} + \frac{1}{5!}) = |x|^5/20$. When $a = 2$, the difference quotient (1) becomes

$$\frac{\frac{1}{3}x^3 + \mathcal{E}}{x^3} \to \frac{1}{3} \qquad \text{when } x \to 0$$

since $|\mathcal{E}/x^3| \le |x|^2/20 \to 0$. Thus $f_2'(0)$ exists and equals $1/3$. When $a < 2$, the difference quotient (1) becomes

$$|x|^{2-a} \cdot \frac{\frac{1}{3}x^3 + \mathcal{E}}{x^3} \to 0 \cdot \frac{1}{3} = 0,$$

so $f_a'(0)$ exists and equals 0.

Part B, for Section 501 only

4. The following table has three missing entries: $f'(1)$, $g'(1)$, and $g'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	1	1		
2	2	1	5	

Determine the missing values if

$$(f \circ g)'(1) = 0,$$
$$(f \circ g)'(2) = 36,$$
$$(g \circ f)'(2) = 45.$$

Solution. Apply the chain rule. From the third condition,

$$45 = (g \circ f)'(2) = g'(f(2))f'(2) = g'(2)f'(2) = g'(2) \cdot 5, \quad \text{so} \quad g'(2) = 9.$$

From the second condition,

$$36 = (f \circ g)'(2) = f'(g(2))g'(2) = f'(1) \cdot g'(2) = f'(1) \cdot 9, \quad \text{so} \quad f'(1) = 4.$$

From the first condition,

$$0 = (f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = 4g'(1), \quad \text{so} \quad g'(1) = 0.$$

Here is the complete table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	1	1	4	0
2	2	1	5	9

5. Give an example of a function $f\colon (0,1) \to \mathbb{R}$ that is increasing, convex, and not uniformly continuous.

Solution. One example is $1/(1-x)$. The derivative is $1/(1-x)^2$, which is positive, so the function is increasing. The second derivative is $2/(1-x)^3$, which is positive when $x < 1$, so the function is convex. The function is unbounded on the bounded interval $(0, 1)$, so the function cannot be uniformly continuous (by the first theorem in Section 5.6).

6. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\lim\limits_{x\to 0} f(x^2)$ exists but $\lim\limits_{x\to 0} f(x)$ does not exist.

Solution. Here is one example:

$$f(x) = \begin{cases} 1, & \text{when } x \ge 0, \\ 0, & \text{when } x < 0. \end{cases}$$

Since x^2 is never negative, $f(x^2)$ is identically equal to 1, so $\lim_{x\to 0} f(x^2)$ exists and equals 1. But $\lim_{x\to 0} f(x)$ does not exist, because the left-hand limit equals 0, while the right-hand limit equals 1.

More generally, any function that has a jump discontinuity at 0 serves as an example.

Part C, for Section 200 only

7. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ for which there are infinitely many real numbers a with the property that $\liminf\limits_{x\to a-} f(x) > \limsup\limits_{x\to a+} f(x)$ (in other words, the limit inferior on the left-hand side exceeds the limit superior on the right-hand side).

Solution. This problem is essentially the same as Exercise 5.3.11 in the textbook. One example is $-\lceil x \rceil$, the negative of the ceiling function. Indeed, if n is an integer, then

$$\liminf_{x\to n-} -\lceil x \rceil = \lim_{x\to n-} -\lceil x \rceil = -n > -(n+1) = \lim_{x\to n+} -\lceil x \rceil = \limsup_{x\to n+} -\lceil x \rceil.$$

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8. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ for which the four Dini derivates at the origin all have different values from each other.

   Solution. Here is one example:

$$f(x) = \begin{cases} x \sin(1/x), & \text{when } x > 0, \\ 0, & \text{when } x = 0, \\ 2x \sin(1/x), & \text{when } x < 0. \end{cases}$$

   The upper and lower right-hand Dini derivates are 1 and -1, while the upper and lower left-hand Dini derivates are 2 and -2.

9. Show that if $f : \mathbb{R} \to \mathbb{R}$ is convex, and f' (the first derivative) exists everywhere, then f' is necessarily continuous.
   Hint: Can a derivative ever have a jump discontinuity?

   Solution. Near a jump discontinuity, the intermediate-value property evidently fails to hold. But derivatives always have the intermediate-value property (Darboux's theorem). Consequently, a derivative cannot have a jump discontinuity.

   If f is convex, and f' exists, then f' is monotonic (nondecreasing); see Corollary 7.35. A monotonic function has one-sided limits at all points of its domain. Hence the only possible discontinuities of monotonic functions are jump discontinuities. (See Section 5.9.2.)

   The first paragraph says that f' has no jump discontinuities. The second paragraph says that if f' has any discontinuities, they must be jump discontinuities. Putting the two conclusions together shows that f' has no discontinuities. In other words, f' is a continuous function.

   This problem is Exercise 7.10.6 in the textbook.