Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Consider the quaternion group $\{1, -1, i, -i, j, -j, k, -k\}$. Determine all the distinct (left) cosets of the subgroup $\{1, -1, i, -i\}$.

   Solution. Method 1. The subgroup $\{1, -1, i, -i\}$ itself is the coset of the identity. Multiplying the elements of the subgroup by j gives a second coset, $\{j, -j, -k, k\}$ (since $ji = -k$). Multiplying the elements of the subgroup by k gives the same second coset, but with the elements presented in a different order.

   Method 2. Since the whole group has eight elements, while the subgroup has four elements, there must be exactly two distinct cosets: the cosets partition the group into disjoint subsets of equal cardinality. Since the subgroup itself is one coset, the other coset must consist of all the remaining elements: namely, $\{j, -j, k, -k\}$.

2. Let $S(5)$ be the symmetric group [consisting of all permutations of the set $\{1, 2, 3, 4, 5\}$], and let H be the cyclic subgroup generated by the permutation (1 2)(3 4 5) [this permutation is written in cycle notation as the product of two disjoint cycles]. How many distinct (left) cosets does the subgroup H have in $S(5)$?

   Solution. You know that the order of a product of disjoint cycles is the least common multiple of their lengths. Therefore the cyclic subgroup H has order 6. The whole group $S(5)$ has order 5!, or 120. By Lagrange's theorem, the number of left cosets of H equals 120/6, or 20.